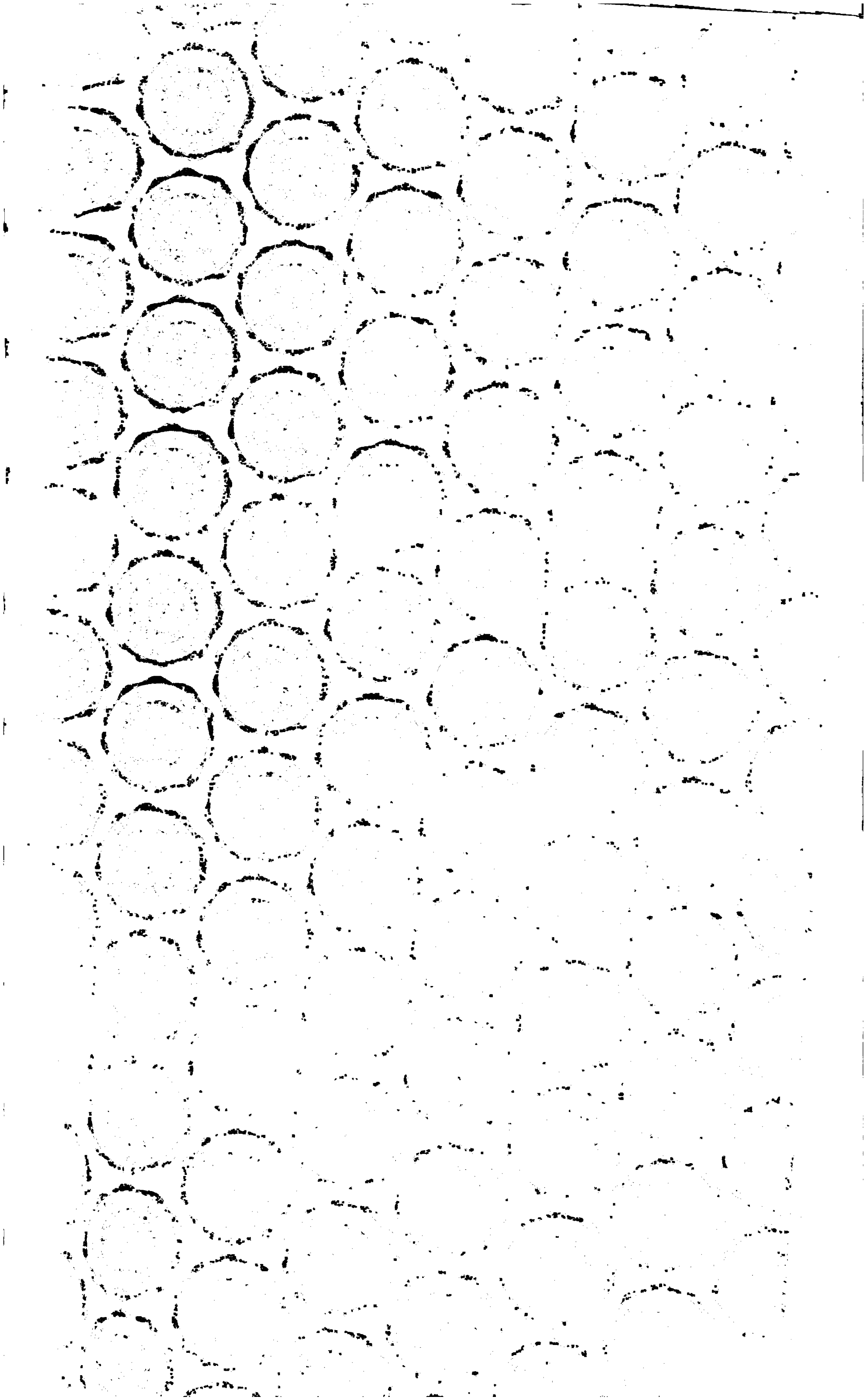

A COURSE OF PLANE GEOMETRY FOR ADVANCED STUDENTS

PART II.

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A COURSE OF PLANE GEOMETRY

FOR

ADVANCED STUDENTS

PART II.

BY

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PREFACE

THE first part of this treatise, published a year ago, dealt with the geometry of the straight line and circle; the present volume contains a moderately comprehensive treatment of the geometry of the conic.

The author holds very strongly that the educational value of higher geometry lies in that novelty of idea and generality of conception which is more characteristic of this subject than of any other branch of elementary mathematics. A course of geometrical conics, based on the focus-directrix definition and developed by Euclidean methods, provides excellent practice in rider-work; but although the student may gain from it a systematic knowledge of the metrical properties of the conic, yet he will assimilate few, if any, really new ideas. On the other hand a projective treatment introduces the student to a region of geometrical thought, unlike anything he has seen in the past—a transition as abrupt and fertile as the crossing from algebra to the calculus. There are constant surprises, apparent contradictions, features of absorbing interest, and principles which, by the generality of their application and the variety of their expression, cannot fail to fascinate the reader and incite him to investigate their developments for himself. Experience proves incontestably that analytical methods frequently elucidate difficult geometrical conceptions. The theory of ideal elements in pure geometry, the notion of one-to-one correspondence and its application to homography and involution, the principles of conical projection are undoubtedly illuminated by a joint use of geometry and analysis.

In the case of imaginary elements, so much simplicity is secured, without any loss of rigour, by adopting an analytical basis that a purely geometrical treatment has been entirely omitted in the

following pages. For, although the construction of a geometrical theory of imaginaries is of remarkable theoretical interest, yet the intrinsic difficulties render it unsuitable for an elementary text-book. The subject-matter has been arranged with a view to enabling the reader to understand the principles of general projection for real and imaginary elements at as early a stage as possible; not only because it furnishes him with an invaluable weapon for attacking a large class of problems, but also because, to a greater extent than in the case of inversion and reciprocation, it enables him to discover properties for himself—an exercise which is both stimulating and instructive.

As in Part I., the number of riders appended to each theorem or group of theorems is very considerable, far more of course than will be required by any single student. If however the book is being used with a class, the significance of a particular theorem can frequently be emphasised by taking a sequence of dependent riders very rapidly on the black-board, asking for suggestions, and so working through half a dozen or more simple applications in a few minutes. This certainly forms a useful variation to the slow progress made when the student works alone and unaided, although it cannot be regarded as a substitute; and in order that such a method may be rendered possible, an ample supply of examples is essential.

The general scheme here adopted has grown out of an attempt to present the subject in a compact and elementary form for school purposes, and has been modified in a variety of ways suggested by experience; but, at the same time, the author is profoundly conscious of all that he owes both to friends and books. Those who are acquainted with Chasles' powerful and original text-books on the Conic Sections and Homography will recognise the wide measure of indebtedness to that source. And it is obvious that there must be a very real dependence on the standard treatises of Cremona, Reye, Russell, Salmon, and Taylor; while the writings of Askwith, Casey, Duporcq, Filon, and Macaulay have also been consulted. A number of historical notes have been inserted, the material of which is mainly derived from Chasles' *Aperçu Historique*, but also from the works of Rouse Ball, Cajori, Heath and Taylor. The author has received a number of valuable suggestions from Mr. G. H. Hardy and Dr. A. N. Whitehead; the first chapter has been read and criticised by

Mr. C. Godfrey and Mr. H. W. Richmond; and Mr. A. E. Broomfield has with great kindness written the entire section on practical solid geometry. The arduous task of reading the whole of the proofs has been undertaken most generously by Mr. R. M. Wright, whose keen sense of logic has effected many improvements of matter and style; and the table of contents and index have been constructed by Mr. G. A. Herman. The Editor of the *Educational Times* has courteously authorised the use of problems which have appeared in the columns of that journal and the Syndics of the Cambridge University Press and the Clarendon Press have kindly accorded their permission for the inclusion of questions set in recent College and University papers.

WINCHESTER,
January, 1910.

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CHAPTER I.

ANALYTICAL IDEAS IN PLANE GEOMETRY.

THE application of algebraic methods to geometrical research by Descartes (1596–1650) opened out an entirely new field of view, and thereby gave a wonderful impetus to the progress of mathematical discoveries. It laid bare certain general principles, which correlate a large number of apparently dissimilar theorems: and so consolidated in method and enunciation the isolated achievements of independent thinkers. Moreover it supplied a sound basis for further research, by expressing in obtrusive analytical terms the fundamental laws governing the structure of lines and curves.

In order to enunciate analytical theorems with the utmost generality of which they are capable, it is necessary to enlarge the conceptions in which they are to find expression. The notion of **number**, starting from the instinctive idea of positive integers, was gradually extended so as to embrace successively fractional, negative, irrational, and finally complex numbers, required respectively for the solution of the equations:

$$ax = b; \quad ax + b = 0; \quad ax^2 = b; \quad (ax + b)^2 + c = 0;$$

where a, b, c represent any positive integers whatever. With this notion of number, it was then possible—but not before—to formulate the **general** theorem, that every equation of degree n is satisfied by n numbers.

This instance serves to illustrate the principle that valuable generalisations are frequently rendered possible by the removal of certain (possibly unconscious) limitations, which had been previously imposed. Thus, if complex numbers are regarded as inadmissible, it is no longer true to say that every equation has a root. Now it may be urged that it is easy to form a clear

conception of numbers such as 2, -3, $\frac{2}{3}$, $\sqrt{5}$, but that a number such as $\sqrt{-3}$ is wholly unintelligible. The reason why we tend to call numbers of the first kind **real**, and of the second kind **imaginary** or **impossible**, is almost entirely due to the fact that we are accustomed to represent graphically the former type, and do not readily associate any graphical connection with the latter type. Anyone by drawing the hypotenuse of a right-angled triangle, the sides of which are 1 inch, 2 inches, finds it easy to accept the existence of $\sqrt{5}$. The graphical representation of complex numbers, furnished by vector geometry, is less familiar; and consequently the difficulties, presented by imaginary elements, are more considerable. But, in point of fact, such geometrical assistance is wholly immaterial. The existence of a number does not depend on whether or not it can be represented under some geometrical form. It may be easier to understand its nature—as it is certainly easier to explain its properties—if it is capable of graphical representation. But its use and its meaning—its organism—depend solely on the laws and operations to which it is subject.

Now complex numbers obey the same fundamental laws of Algebra as real numbers: and may therefore enter with equal freedom into analytical processes. If the result of any sequence of such processes is capable of expression in geometrical language, then its enunciation in geometrical terms is valid, even if any part of the process cannot be represented graphically. For since the process is valid analytically, its conclusions must be true.

It is however of value to widen the basis of geometrical reasoning by borrowing from analysis notions which are incapable of graphical depiction. This procedure is usually referred to as the **Principle of Continuity.*** The germ of this principle dates as far back as the time of Kepler (1571–1630), who recognised, without the aid of analysis, that the different species of conics were not isolated curves, having each a geometry peculiar to itself, but formed a continuous chain, the ellipse passing into the parabola, and the parabola into the hyperbola. The first complete exposition was given by Boscovich (1711–1787), in what was intended to form an elementary school text-book on conic sections. But it was Poncelet's (1788–1867) great discovery of the "circular points at infinity" and their connection with the foci of a conic that revealed,

* Much interesting historical information will be found in Dr. Taylor's *Ancient and Modern Geometry of Conics*.

in all its stimulating generality, the wide range of application of this principle.

It is worth while considering at this stage the analytical bearing of the Principle of Continuity.

The straight line $x \cos \alpha + y \sin \alpha = p$ cuts the circle $x^2 + y^2 = a^2$ at the two points, whose coordinates are

$$(p \cos \alpha \pm \sin \alpha \sqrt{a^2 - p^2}, \quad p \sin \alpha \mp \cos \alpha \sqrt{a^2 - p^2}).$$

These coordinates are real if $p < a$;
coincident if $p = a$;
complex if $p > a$.

By making use of the enlarged conception of **number**, it is possible to enunciate the **general** theorem, that every straight line meets every circle at two "points." This attaches a new significance to the term "point," by removing a limitation, previously imposed, that points must be real. Any pair of real numbers, *e.g.* $x = 3$, $y = 4$, can be represented graphically by the position of a point referred to real coordinate axes, and so can be associated and identified geometrically with a point. If either or both the numbers of a given pair are complex, for the sake of continuity of expression, the pair is still identified with a point, called an **imaginary point**, and necessarily incapable of direct graphical representation. The usage of the term is merely conventional, and its importance is simply due to the fact that it corresponds to a definite analytical feature. In the same way, the circle $x^2 + y^2 = a^2$, which is the locus of all "points" at distance a from the origin, does not merely include such points as appear in the figure, but all points which correspond to a pair of numbers, real or complex, satisfying that equation.

Again, the aggregate of pairs of values of x, y which satisfy the equation $ax + by + c = 0$, where a, b, c are any constant numbers, constitutes the straight line associated with that equation. A straight line may therefore contain only one point, or no points at all, capable of being represented on paper; *e.g.* the line $x - 2 + \sqrt{-1}(y - 3) = 0$ contains only one real point $(2, 3)$, and the line $x + y = \sqrt{-1}$ contains no real **finite** point.

By extending the meaning of geometrical terms, it is thus possible to enunciate theorems with greater generality. But in addition to this, it may enlarge the range of application of a proof, which applies apparently only to a special case. For example, if two

circles intersect at real points, it is easy to show that the locus of a point, whose powers* w.r.t. the two circles are equal, is a straight line, viz. the common chord. This is expressible in analytical terms, and since the analytical method of proof takes no account of whether the points of intersection are real or imaginary, it follows that the theorem must be true in general. In other words, the proof of the general case is involved in, and consequent from the proof of the special case.

Free use of imaginary points and lines may therefore be made, in order to generalise the range of application of geometrical theorems; and this is referred to under the name of the **Principle of Continuity**. The application of the idea of a limit in geometry is another feature of this principle, bearing indeed more directly on Kepler's line of thought; but it would be out of place to develop it here.

EXTENSIONS OF GEOMETRICAL TERMS.

For brevity, $+\sqrt{-1}$ or $\left(1, \frac{\pi}{2}\right)$ will in future be denoted by i.

Definitions.

(1) Any pair of numbers, real or complex, are said to be represented by, and identified with a **Point**, in a given plane.

(2) Any aggregate of pairs of numbers, real or complex, which satisfy the equation $ax+by+c=0$, where a, b, c are any constant numbers, real or complex, are said to constitute a **Straight Line** in a given plane.

In general, if any or all the constants a, b, c are complex, the straight line is imaginary; otherwise, the line is real.

(3) If the coordinates of two points are conjugate imaginaries, *i.e.* $(a+ia', b+ib')$; $(a-ia', b-ib')$, the points are called **conjugate imaginary points.**

(4) If the constant coefficients in the equations of two lines are respectively conjugate imaginaries, *i.e.* $(a+ia')x+(b+ib')y+c+ic'=0$; $(a-ia')x+(b-ib')y+c-ic'=0$, the two lines are called **conjugate imaginary lines.**

Unless otherwise stated, it will be assumed that the equations of all conics mentioned are real (*i.e.* the coefficients are real).

* The power of a point P w.r.t. a circle centre A, radius a, is defined as PA^2-a^2.

From the definition of a straight line, it follows that any two straight lines $a_1x+b_1y+c_1=0$; $a_2x+b_2y+c_2=0$ meet at one and only one point. The apparent exception, where $\frac{a_1}{b_1}=\frac{a_2}{b_2}$, *i.e.* the case of parallelism, will be dealt with later.

1. Does a real line contain many, one, or no imaginary points?

2. Does an imaginary line contain many, one, or no real points?

3. Is it possible to draw any, one, or many real lines through a given imaginary point?

4. Is it possible to draw any, one, or many imaginary lines through a given real point?

THEOREM 1.

Two conjugate imaginary lines meet at a real point.

Any two conjugate imaginary lines can be represented by

$ax+by+c+i(a'x+b'y+c')=0$; $ax+by+c-i(a'x+b'y+c')=0$,

where a, b, c, a', b', c' denote real constants.

These meet at the point given by the two real equations

$$ax+by+c=0; \quad a'x+b'y+c'=0,$$

which is necessarily real. Q.E.D.

THEOREM 2.

Two conjugate imaginary points lie on a real line.

The proof is left to the reader.

[Take $(x_1+ix_2,\ y_1+iy_2)$; $(x_1-ix_2,\ y_1-iy_2)$ as the two points.]

THEOREM 3.

Every imaginary line contains one and only one real point.

Any imaginary line can be represented by

$$ax+by+c+i(a'x+b'y+c')=0.$$

This is satisfied by the pair of real values of x, y given by

$$ax+by+c=0; \quad a'x+b'y+c'=0.$$

$\therefore$ it contains one real point.

Further, if x, y are real, the two equations $ax+by+c=0$; $a'x+b'y+c'=0$ must be satisfied.

$\therefore$ there is only one real point. Q.E.D.

THEOREM 4.

Every imaginary point lies on one and only one real line.

The proof is left to the reader.

THEOREM 5.

Every real line contains an unlimited number of imaginary points, which occur in conjugate pairs.

Giving x any complex value whatever, there corresponds from the equation a definite value of y, and this pair of values determines an imaginary point. Hence the number of imaginary points on the line is unlimited.

Further, let $(x_1+ix_2,\ y_1+iy_2)$ be one imaginary point on $ax+by+c=0$; a, b, c being real by hypothesis.

Then $$ax_1+by_1+c+i(ax_2+by_2)=0;$$

$$\therefore\ ax_1+by_1+c=0 \text{ and } ax_2+by_2=0;$$

$$\therefore\ ax_1+by_1+c-i(ax_2+by_2)=0;$$

$\therefore$ the point $(x_1-ix_2,\ y_1-iy_2)$ also lies on the line;

$\therefore$ all imaginary points on the line occur in conjugate pairs.

Q.E.D.

THEOREM 6.

Through every real point there pass an unlimited number of imaginary lines; and these occur in conjugate pairs.

The proof is left to the reader.

THEOREM 7.

If two lines l_1, l_2 meet at a point P, and if the conjugate imaginary lines l_1', l_2' meet at P', then P and P' are conjugate imaginary points.

To find the coordinates of P, we solve the equations of l_1, l_2. If we now write $-i$ for i in this work, we have the solution of the equations of l_1', l_2', giving P'. Hence the coordinates of P' are derived from the coordinates of P by writing $-i$ for i.

$\therefore$ P and P' are conjugate imaginary points. Q.E.D.

THEOREM 8.

(1) A real line meets a real conic in two points, which are real or conjugate imaginaries.

(2) Two conics intersect at four points, which are either all real, or two real and two conjugate imaginaries, or two pairs of conjugate imaginaries.

The proof is left to the reader.

[Use analysis and note that the roots of a quadratic or quartic, with real coefficients, are real or occur in conjugate pairs.]

THEOREM 9.

Two conjugate imaginary lines meet a real conic in two pairs of conjugate imaginary points.

The proof is left to the reader.

5. Prove Theorem 2.

6. Prove Theorem 4.

7. Prove Theorem 6.

8. Prove Theorem 8.

9. Prove Theorem 9.

10. Find the equation of the real line passing through (1) $3,\ 4+i$; (2) $2+3i,\ 3-2i$.

11. Find the real point on the line $(2+3i)x+(3-2i)y=8-i$.

12. If the vertices of a quadrangle are two pairs of conjugate imaginary points, prove that the diagonal point triangle is real.

13. If two of the vertices of a quadrangle are real points, and if the other two are conjugate imaginaries, prove that one vertex and the opposite side only, of the diagonal point triangle are real.

14. Enunciate and prove the dual of Ex. 12.

15. Enunciate and prove the dual of Ex. 13.

16. A quadrilateral is formed by drawing the tangents from two conjugate imaginary points to a given real conic, and the diagonals are drawn: determine which points and lines in the figure are real, and which are conjugate imaginaries.

17. What does the equation $x^2+y^2=0$ represent?

18. What do the equations

$$(1)\ x^2+y^2-6x-8y+21=0;\quad (2)\ x^2+y^2-6x-8y+25=0;$$
$$(3)\ x^2+y^2-6x-8y+29=0;$$

represent?

19. P is a variable imaginary point on a fixed real line; prove analytically that the real point on the polar of P w.r.t. a given real circle is a fixed point.

If the straight lines $y = m_1x + c_1$; $y = m_2x + c_2$ are both real, it is easy to prove that the tangent of the angle between them is represented by $\frac{m_1 - m_2}{1 + m_1m_2}$. But, if either line is imaginary, the term 'angle' ceases to have any meaning. In order to secure continuity of statement, it is therefore convenient to make some definition which will apply to this case. The essential condition, for this purpose, is that there should be no analytical distinction between the two meanings ascribed to the word 'angle.' But it is further desirable that the definition employed should never lead to geometrical interpretations which are contrary to recognised general theorems.

Definition.

If either or both the lines $y = m_1x + c_1$; $y = m_2x + c_2$ are imaginary, the expression $\frac{m_1 - m_2}{1 + m_1m_2}$ is defined to be the tangent of the **angle** between the two lines, provided that this expression, for any constant value of either of the quantities m_1, m_2, always takes different values for different values of the other.

We will first consider the meaning and desirability of the proviso which enters into this definition.

Suppose that $\frac{m_1 - m_2}{1 + m_1m_2} = k$ for two different values of m_1.

Since the equation $m_1(1 - km_2) - (k + m_2) = 0$ is of the first degree in m_1 and is satisfied by two values of m_1, the coefficient of m_1 is zero;

$$\therefore \; 1 - km_2 = 0 \quad \text{and} \quad k + m_2 = 0;$$

$$\therefore \; m_2^2 = -1 \quad \text{or} \quad m_2 = \pm i \quad \text{and} \quad k = \mp i.$$

Hence, if $m_2 = \pm i$, the value of the expression is equal to $\mp i$ for **all values** of m_1. But in every other case it takes different values for different values of m_1.

The last definition may therefore be stated as follows:

Definition.

If either or both the lines $y = m_1x + c_1$; $y = m_2x + c_2$ are imaginary, the expression $\frac{m_1 - m_2}{1 + m_1m_2}$ is defined to be the tangent of the **angle**

between the two lines, provided that neither m_1^2 nor m_2^2 is equal to -1.

According then to our definition, the term 'angle' must not be applied to the case where either of the lines is of the form $y = \pm ix + c$.

It is easy to see why this case has to be excluded. Suppose that it is proved that two lines OP, OP' make equal angles in the same sense (*i.e.* angles whose tangents are equal) with a third line OA, then we say that the lines OP, OP' coincide. But this ceases to be true if the line OA is of the form $y = \pm ix + c$, for the tangent of the 'angle' between $y = mx + d$, $y = ix + c$ would appear to be $\dfrac{m-i}{1+mi} = \dfrac{-i(1+mi)}{1+mi} = -i$, which does not depend on the value of m.

In other words, it is impossible to associate any idea of direction with the lines $y = \pm ix + c$. Any argument, which is based on the direction of these lines, is essentially invalid. It is of such importance to understand this, that it is worth while illustrating the same idea by a further and somewhat striking example:

If the lines $y = m_2 x$, $y = m_3 x$ make angles θ_{12}, θ_{13} with the line $y = m_1 x$, then $y = m_2 x$ makes with $y = m_3 x$ an angle θ_{32} given by $\tan\theta_{32} = \dfrac{\tan\theta_{12} - \tan\theta_{13}}{1 + \tan\theta_{12}\tan\theta_{13}}$ unless $m = \pm i$, in which case the last expression is indeterminate.

By definition,

$$\begin{aligned}\tan\theta_{12} - \tan\theta_{13} &= \frac{m_2 - m_1}{1 + m_1 m_2} - \frac{m_3 - m_1}{1 + m_3 m_1}\\ &= \frac{m_2 - m_1 + m_1 m_2 m_3 - m_1^2 m_3 - m_3 + m_1 - m_1 m_2 m_3 + m_1^2 m_2}{(1 + m_2 m_1)(1 + m_3 m_1)}\\ &= \frac{(m_2 - m_3)(1 + m_1^2)}{(1 + m_2 m_1)(1 + m_3 m_1)},\end{aligned}$$

and $\quad 1 + \tan\theta_{12}\tan\theta_{13}$

$$\begin{aligned}&= \frac{1 + m_2 m_1 + m_3 m_1 + m_1^2 m_2 m_3 + m_2 m_3 - m_1 m_3 - m_1 m_2 + m_1^2}{(1 + m_2 m_1)(1 + m_3 m_1)}\\ &= \frac{(1 + m_2 m_3)(1 + m_1^2)}{(1 + m_2 m_1)(1 + m_3 m_1)};\end{aligned}$$

$$\begin{aligned}\therefore\ \frac{\tan\theta_{12} - \tan\theta_{13}}{1 + \tan\theta_{12}\tan\theta_{13}} &= \frac{(m_2 - m_3)(1 + m_1^2)}{(1 + m_2 m_3)(1 + m_1^2)}\\ &= \tan\theta_{32}, \text{ unless } 1 + m_1^2 = 0.\end{aligned}$$

If however $1+m_1^2=0$ or $m_1=\pm i$, the expression becomes indeterminate. [The case where $\tan\theta_{12}$ or $\tan\theta_{13}$ increases without limit, *i.e.* $1+m_2m_1=0$ or $1+m_3m_1=0$ has been neglected: but could easily be treated separately.]

This example shows that the addition theorem for tangents of angles would cease to be always true if the proviso, introduced into the definition, were omitted. Now it is essential that all general formulae should remain true, independently of whether any of the quantities are real or complex. And therefore, just because the notion of an angle is being generalised to secure continuity between the domains of 'real' and 'imaginary' elements, it is of paramount importance that this limitation should be introduced into the definition adopted.

In the special case where one of the lines is the x-axis, $y=0$, the line $y=mx+c$ makes an angle θ with it, given by

$$\tan\theta=\frac{m-0}{1+m\times 0}=m.$$

It is worth perhaps pointing out in conclusion that the line $y=\pm ix+c$ would make with the x-axis, if the proviso were omitted, an angle θ given by $\tan\theta=\pm i$, which, pursuing the same false supposition, is the angle of an imaginary right-angled triangle, whose sides are 1, i and whose hypotenuse is $\sqrt{1+i^2}=\sqrt{1-1}=0$.

This graphical illustration may perhaps convince the reader of the innumerable difficulties which would occur, if any attempt were made to attach a meaning to the symbol $\tan^{-1}(i)$.

Definition.

The lines $y=\pm ix+c$ are called the **isotropic** lines. This name is due to Laguerre.

20. Prove that the equation of the line $y=ix$ is unaltered when the axes are rotated through any angle θ, the origin remaining unaltered. Interpret this geometrically.

21. Assuming that the ordinary formula for the distance between two real points still holds good, prove that the distance between any two points on the line $y=ix$ is zero. Interpret this geometrically.

22. If $\tan\theta=i$, what would be the values of $\sin\theta$, $\cos\theta$, according to the usual formulae?

23. Prove that the formula, $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$, is still valid, if θ is the angle between two imaginary lines.

Starting from the general definition of the tangent of an angle, given above, and adding to it similar definitions for the sine, cosine, etc., there would be no difficulty in establishing all the usual trigonometrical formulae, obtained for real angles (*e.g.* see p. 9). But for the present purpose it is unnecessary to enter into further details.

CROSS RATIO.

Let l_1, l_2, l_3, l_4 be four real concurrent lines: denote by θ_{12} the angle l_2 makes with l_1, etc. Then the cross ratio of the pencil formed by these four lines is $\dfrac{\sin\theta_{12}\,.\sin\theta_{34}}{\sin\theta_{14}\,.\sin\theta_{32}}$. (Part I., p. 103.) And these four lines form a harmonic pencil, if this expression $= -1$. If any of the four lines are imaginary, it is necessary to make a new definition, since the term cross ratio as applied to imaginary lines has not any direct significance. It might appear at first sight simplest to define it by means of the angular expression, which is the characteristic for four real lines. But this would have the extreme inconvenience of excluding the isotropic lines from the idea of a pencil.

Suppose that the four lines are respectively, referred to their vertex as origin,

$$y - m_1x = 0;\quad y - m_2x = 0;\quad y - m_3x = 0;\quad y - m_4x = 0.$$

Then $\quad \tan\theta_{12} = \dfrac{m_2 - m_1}{1 + m_1m_2};\quad \therefore\ \sin\theta_{12} = \dfrac{m_2 - m_1}{\sqrt{(1 + m_1^2)(1 + m_2^2)}};$

$$\therefore\ \text{the cross ratio} = \frac{(m_2 - m_1)(m_4 - m_3)}{(m_4 - m_1)(m_2 - m_3)},$$

which is now in a form which applies to every case.

Definition.

If four concurrent lines, referred to their common vertex as origin, are represented by $y = m_1x$; $y = m_2x$; $y = m_3x$; $y = m_4x$; the expression $\dfrac{(m_2 - m_1)(m_4 - m_3)}{(m_4 - m_1)(m_2 - m_3)}$ is called the **cross ratio** of these four lines (taken in the order stated above): and if the value of the cross ratio is -1, the four lines are said to form a **harmonic pencil**: and the pairs of lines $y = m_1x$, $y = m_3x$; $y = m_2x$, $y = m_4x$ are said to be **harmonically conjugate** to each other.

If all the lines are real, the above form of the cross ratio is

equivalent to the angular form originally stated: and this equivalence still remains, when any or all the lines are imaginary, provided that none of the lines are isotropic.

In order to enunciate the generalised metrical properties of ranges and pencils, it would be necessary first of all to define what is meant by the length of the line joining two imaginary points. It would then be easy to deduce the usual theorems. But this will be left to the reader. (See Ex. 24.)

24. Formulate a definition for the length of the line joining two imaginary points. What proviso should be inserted?

25. The coordinates of A, B, C are $(0, 0)$, $(3, 3i)$, $(-2, 2i)$ respectively; prove that $(AB+AC)^2$ is less than BC^2, according to the ordinary analytical formulae. What theorem does this violate?

26. Calculate the cross ratio of the lines

$$y-mx=0,\quad y-ix=0,\quad my+x=0,\quad y+ix=0.$$

State the result in the form of a general theorem.

27. Calculate the cross ratio of the lines

$$y-m_1x=0,\quad y-ix=0,\quad y-m_2x=0,\quad y+ix=0;$$

and prove that it is constant, if $\dfrac{m_1-m_2}{1+m_1m_2}$ is constant.

State the result in the form of a general theorem.

THEOREM 10. [LAGUERRE'S THEOREM.]

If α is the angle between the lines $y=mx$, $y=m'x$, then the cross ratio of the pencil formed by the lines $y=mx$, $y=ix$, $y=m'x$, $y=-ix$, is equal to $\cos 2\alpha+i\sin 2\alpha$ or $exp(2\alpha i)$.

Let $m=\tan\theta$, $m'=\tan\theta'$, so that $\alpha=\theta-\theta'$.

The cross ratio

$$=\frac{(i-m)(-i-m')}{(-i-m)(i-m')}=\frac{(1+im)(1-im')}{(1-im)(1+im')}$$

$$=\frac{(\cos\theta+i\sin\theta)(\cos\theta'-i\sin\theta')}{(\cos\theta-i\sin\theta)(\cos\theta'+i\sin\theta')}=\frac{\cos(\theta-\theta')+i\sin(\theta-\theta')}{\cos(\theta-\theta')-i\sin(\theta-\theta')}$$

$$=\cos 2\alpha+i\sin 2\alpha=exp(2\alpha i).$$

Q.E.D.

In particular, if $\alpha=\frac{\pi}{2}$, the cross ratio $=-1$.

This theorem may now be stated geometrically, by using terms in their wider significance.

THEOREM 11.

(1) Any pair of perpendicular lines are harmonically conjugate to the isotropic lines.

(2) If a variable pair of lines contain a constant angle, they form with the isotropic lines a pencil of constant cross ratio.

(3) If two lines are harmonically conjugate to the isotropic lines, they must be perpendicular to each other.

(4) If a variable pair of straight lines form with the isotropic lines a pencil of constant cross ratio, they must make a constant angle with each other.

THEOREM 12.

The isotropic lines are the only lines which are harmonically conjugate to each of two pairs of perpendicular lines, $y = mx$, $y = m'x$ and $y = lx$, $y = l'x$.

Let each pair be harmonically conjugate to $y = \lambda x$, $y = \mu x$.

By definition, $(m - \lambda)(m' - \mu) + (m' - \lambda)(m - \mu) = 0$;

$$\therefore \quad 2mm' + 2\lambda\mu - (\lambda + \mu)(m + m') = 0;$$

$$\therefore \quad 2(\lambda\mu - 1) = (\lambda + \mu)\left(m - \frac{1}{m}\right), \text{ since } mm' = -1.$$

Similarly,
$$2(\lambda\mu - 1) = (\lambda + \mu)\left(l - \frac{1}{l}\right);$$

$$\therefore \quad 0 = (\lambda + \mu)\left(m - l + \frac{m - l}{lm}\right);$$

$$\therefore \quad 0 = (\lambda + \mu)(m - l)(lm + 1);$$

If $l = m$, the two line pairs are coincident.

If $l = -\dfrac{1}{m} = m'$, the two line pairs are coincident.

$$\therefore \quad \lambda + \mu = 0; \quad \therefore \quad \lambda\mu - 1 = 0;$$

$$\therefore \quad \lambda^2 = \mu^2 = -1;$$

$\therefore$ the only line pair is the pair of isotropic lines.

Q.E.D.

It is owing to these properties, and the additional fact that the value of a cross ratio is unaffected by any homographic transformation, to which we next proceed, that the isotropic lines play so important a part in projective geometry.

TRANSFORMATION.

It is often possible to deduce from one known property another property, apparently of an entirely different character, by effecting a correspondence between two distinct geometrical systems. Two examples of this process of transformation have already been met with.

It has been pointed out [Part I., pp. 110, 163] how, from a given system of lines and points, it is possible to build up a corresponding system of points and lines by the method known as the Principle of Duality. To any point A of the first system corresponds one and only one line a of the second: and to any line a of the second corresponds one and only one point A of the first. Between two such systems there exists what is called a **one-to-one** or (1, 1) correspondence.

Another method of transformation was exhibited by the process of Inversion. In this case (see p. 21), to any point A of the first system (excluding the origin) corresponds one and only one point A' of the second; and conversely. But, in general, straight lines in one system do not correspond to straight lines in the other. Analytically, if (x, y); (ξ, η) are two corresponding points in the two systems, the transformation is effected by

$$\xi = \frac{k^2 x}{x^2+y^2}, \quad \eta = \frac{k^2 y}{x^2+y^2};$$

where k is a constant (the radius of inversion).

We shall now discuss a (1, 1) correspondence in which points correspond to points and straight lines to straight lines. This is illustrated by the method of orthogonal and conical projection, the latter of which is one of the chief subjects of discussion in the present volume.

Consider any figure C, consisting of points, lines, or curves situated in a plane Σ. Lines are drawn from a fixed point O outside Σ to all these points, and to all the points on the lines or curves of C. These lines, radiating from O, are cut by any other plane Σ' in a system of points, lines, or curves forming a figure C'. Then C' is said to be formed from C by conical projection: and O is called the vertex of projection. Now suppose that both systems C, C' are placed in the same plane: then, owing to the way in which C' has been generated from C, there exists a one-to-one correspondence between C and C'; to every point and

line of C there corresponds one and only one point and line respectively of C', and conversely.

Definition.

Any transformation by which two coplanar figures are so related that any point and line of one correspond to one and only one point and line respectively of the other, and conversely, is called a **homographic transformation.**

To determine the general analytical expression for a homographic transformation.

Let (x, y) be the coordinates of a point A in the first system; and let (ξ, η) be the coordinates of the corresponding point A' in the second (coplanar) system.

It is required to express ξ, η each in terms of x, y.

Now, since the correspondence must be (1, 1), the required expressions cannot contain any radicals: reducing them to a common denominator, we therefore have

$$\xi = \frac{f_1(x, y)}{\phi(x, y)}, \quad \eta = \frac{f_2(x, y)}{\phi(x, y)},$$

where f_1, f_2, ϕ represent polynomials in x, y*; and without any loss of generality, we may suppose that there is no factor common to all three functions f_1, f_2, ϕ.

To any straight line $p\xi + q\eta + r = 0$ corresponds

$$pf_1(x, y) + qf_2(x, y) + r\phi(x, y) = 0,$$

which must be a straight line for all values of p, q, r, by hypothesis.

Therefore, since f_1, f_2, ϕ have no common factor, each must be linear in x, y;

$$\therefore \ \xi = \frac{a_1x + b_1y + c_1}{lx + my + n}, \quad \eta = \frac{a_2x + b_2y + c_2}{lx + my + n}.$$

And it is easy to see, by solving for x, y in terms of ξ, η, that to any line in the (x, y) system there corresponds one and only one line in the (ξ, η) system.

The above expression is therefore the general transformation required: and the letters a_1, b_1, c_1, a_2, b_2, c_2, l, m, n represent any constant quantities, real or imaginary. Q.E.D.

Since the transformation contains eight **independent** constants, it follows that they can be chosen so that any four points, no three of which are collinear, or any four lines, no three of which

* It is assumed that f_1, f_2, ϕ are not transcendental functions.

are concurrent, can be made to correspond to any four points or any four lines, subject to the same conditions.

THEOREM 13.

The cross ratio of any pencil of four concurrent lines is unaltered in value by any homographic transformation.

Take the origin at the point of concurrency, so that the four lines of the ξ, η system may be written: $\eta=m_1\xi$, $\eta=m_2\xi$, $\eta=m_3\xi$, $\eta=m_4\xi$.

With the usual notation, the line corresponding to $\eta=m_1\xi$ is

$$a_2x+b_2y+c_2=m_1(a_1x+b_1y+c_1)$$

or

$$(a_2-m_1a_1)x+(b_2-m_1b_1)y+c_2-m_1c_1=0,$$

which is parallel to $y=-\dfrac{a_2-m_1a_1}{b_2-m_1b_1}x\equiv M_1x$, say.

Then

$$M_2-M_1=\frac{a_2-m_1a_1}{b_2-m_1b_1}-\frac{a_2-m_2a_1}{b_2-m_2b_1}$$

$$=\frac{(m_2-m_1)(a_1b_2-a_2b_1)}{(b_2-m_1b_1)(b_2-m_2b_1)};$$

$\therefore\ \dfrac{(M_2-M_1)(M_4-M_3)}{(M_4-M_1)(M_2-M_3)}$ reduces at once to $\dfrac{(m_2-m_1)(m_4-m_3)}{(m_4-m_1)(m_2-m_3)}$.

Q.E.D.

Corollary.

If four lines form a harmonic pencil, the four corresponding lines also form a harmonic pencil.

THEOREM 14.

The cross ratio of any range of four collinear points is unaltered in value by any homographic transformation.

The cross ratio of any range is equal to the cross ratio of any pencil having this range as a section: and must therefore be unaltered by any homographic transformation, by Theorem 13.

Q.E.D.

THEOREM 15.

The degree of a curve is unaltered by any homographic transformation.

Let the curve C be of degree n.

Then any straight line l cuts the curve C at n points.

$\therefore$ the corresponding straight line λ must cut the corresponding curve Γ at the corresponding n points.

$\therefore$ Γ is of degree n. Q.E.D.

THEOREM 16.

The class of a curve (*i.e.* the number of tangents that can be drawn from any point to the curve) is unaltered by any homographic transformation.

The proof is left to the reader.

In Chapter IV., a conic will be defined as the projection of a circle (see pp. 88–9 and Th. 51). Since projection is one method of establishing a (1, 1) correspondence, it follows that the conic is a curve which is homographically related to a circle.

THEOREM 17.

Any conic is represented by an equation of the second degree.

Since the equation of a circle is of degree 2, it follows, either from Theorem 15 or by direct substitution, that the equation of any conic can be written in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad \text{Q.E.D.}$$

29. Prove Theorem 16.

30. In any two coplanar figures which correspond homographically, prove that there are in general three (finite) points which are self-corresponding.

31. With the data of Ex. 30, how many straight lines are there, which are self-corresponding?

32. In two homographic systems the points (0, 0), (0, 1), (3, 0), (3, 12) correspond respectively to (0, 0), (0, 1), (2, 2), (3, 1); find the position of the third self-corresponding point.

33. Find the self-corresponding points in the transformation defined by $\xi = \dfrac{x+2y}{x+y-1}$, $\eta = \dfrac{2x+y}{x+y-1}$.

34. Find the general homographic relation connecting two systems which have the three points (0, 0); (0, 1); (2, 0) as self-corresponding: and such that lines joining corresponding points meet at the origin.

35. If the homographic relation is $\xi = \dfrac{x+3y-1}{x+2y+1}$, $\eta = \dfrac{2x-y+1}{x+2y+1}$, prove that all lines in the (ξ, η) system parallel to $\xi + 2\eta = 0$ correspond to lines in the (x, y) system which pass through the point $(-\frac{1}{9}, -\frac{4}{9})$.

36. With the notation of the general homographic relation, prove that any class of parallel lines in the (ξ, η) system corresponds to a class of concurrent lines in the (x, y) system: and that the point of concurrency lies on the line $lx+my+n=0$.

37. Prove that the homographic relation $\xi = \frac{p - \frac{q^2}{a}x}{x}$, $\eta = \frac{qy}{x}$ transforms the parabola $\eta^2 = a\xi$ into a circle.

38. Find the values of p, q in order that the relation $\xi = \frac{\mathbf{px}}{\mathbf{x+q}}$, $\eta = \frac{\mathbf{qy}}{\mathbf{x+q}}$ may transform the general conic

$$\mathbf{a}\xi^2 + 2\mathbf{h}\xi\eta + \mathbf{b}\eta^2 + 2\mathbf{g}\xi + 2\mathbf{f}\eta + \mathbf{c} = 0$$

into a circle.

39. If the homographic relation is $\xi = a_1x + b_1y + c_1$; $\eta = a_2x + b_2y + c_2$, prove that parallel lines correspond to parallel lines.

40. What simple homographic relations will make

$$(1)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ and } (2)\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

correspond to the circle $\xi^2 + \eta^2 = c^2$?

THE LINE AT INFINITY.

In every exact science, it is of primary importance that the meanings of all terms employed should be fixed with absolute precision. A term which is vague, or capable of different interpretations, is more than likely to introduce confusion and error. It is therefore desirable at this stage to consider what precise meaning is to be attached to the word "infinity." To say that infinity is a number larger than any number we can think of or write down, is really totally unintelligible. It is a manifest contradiction to imagine a number which is larger than any number that can be thought of. This, then, at least is true—that **infinity is not a number.**

Consider a series of points A_0, A_1, A_2, ... A_n, ... marked off at unit distance apart, along an unlimited straight line. If we proceed to count the number of points on the line, it is clear that it is impossible ever to complete the process: and accordingly it is said that the number of points is unlimited or infinite. The word "infinite" is here used to describe the **nature** of the process. If we count, for instance, the number of pages in this book, after a time the process is completed and is consequently called "finite." Hence, just as "finite" does not state the number of pages, but only describes the nature of the counting, so the corresponding word "infinite" has equally no reference to the number of points on the line, but only to the character of the counting process. It is easy to state the fundamental distinction between the two

processes. If the process is finite, it is possible to name a number N sufficiently large (in this case 400), such that the process is completed before arriving at N: while if the process is infinite, no matter what number N be named, a stage in the counting will arise which exceeds N. This, then, gives a precise sense to the word "infinite" as applied to a series of objects. It is worth while illustrating it further. Consider the statement: "If n tends to become infinite, then $\sqrt{n}$ tends to become infinite." This is an abbreviated form of saying that, whatever number N be named, there always exists a value of n, such that $\sqrt{n} > N$. It is the order of choice which is important. Your opponent has to choose a number N, which he can take as large as he pleases, and afterwards it lies with you to find a value of n such that $\sqrt{n} > N$. If, whatever he does, you are certain to be able to defeat him, then $\sqrt{n}$ tends to infinity. And this statement may be written more briefly, following Mr. Bromwich, in the form: if $n \to \infty$, then $\sqrt{n} \to \infty$, where ∞* stands for "infinity." The reader is, however, strongly urged to avoid the notation: if $n = \infty$, then $\sqrt{n} = \infty$, for (1) ∞ is not a number, and (2) it is a bare self-contradiction to assert that anything can **equal** infinity.

Consider now the homographic transformation

$$\xi = \frac{a_1x + b_1y + c_1}{lx + my + n}, \quad \eta = \frac{a_2x + b_2y + c_2}{lx + my + n}.$$

To any point A in the (x, y) system, there corresponds one and only one point a in the (ξ, η) system, provided that A does not lie on the line $lx + my + n = 0$. As the point A tends to any point of this line, the corresponding values of ξ, η tend towards infinity: and when A actually falls on this line, ξ, η cease to have any corresponding numerical values. The correspondence between the two systems is marred by this discontinuity, that there exists a single line in the (x, y) figure, to which there is no corresponding line in the (ξ, η) figure. To secure generality of statement, it is desirable to eradicate this exceptional case. This is done by inventing a line which will be called the "**ideal line**" or the "**line at infinity**," and adding this to the (ξ, η) system. This line is essentially fictitious and incapable of graphical representation. It

* The symbol ∞ is due to John Wallis (1616-1703) who was educated at Felsted and Cambridge, and afterwards became Savilian Professor of Geometry at Oxford. He was also the first to introduce negative indices into the theory of Algebra.

is created solely for the purpose of corresponding to the line $lx+my+n=0$ and possesses only such properties as are consequent from this correspondence. The name "line at infinity" is indeed misleading, for (1) it is not a line at all in the ordinary sense, and (2) infinity is not a geographical description, as the phrase suggests.

Further, we conceive a series of "**ideal points**" or "**points at infinity**" as composing the line at infinity, their existence being justified solely by the fact that they fulfil the function of corresponding to the points of the line $lx+my+n=0$.

If we solve the equations of the transformation so as to obtain x and y in terms of ξ, η, we obtain relations of the form

$$x=\frac{\alpha_1\xi+\beta_1\eta+\gamma_1}{\lambda\xi+\mu\eta+\nu},\quad y=\frac{\alpha_2\xi+\beta_2\eta+\gamma_2}{\lambda\xi+\mu\eta+\nu}.$$

By following the previous argument precisely, it now appears that there is no line in the (x, y) system which corresponds to the line $\lambda\xi+\mu\eta+\nu=0$ in the (ξ, η) system. Hence, in order to complete the correspondence, it is necessary to add to the (x, y) system a fictitious line which as before is called the "ideal line" or the "line at infinity." By this final addition, the (1, 1) correspondence between the two systems is now complete, without exception.

Consider now a system of concurrent lines in the (x, y) system which meet at a point A on the line $lx+my+n=0$. They correspond to a system of lines in the (ξ, η) system which have no finite point of intersection, *i.e.* to a system of parallel lines. But since A corresponds to a single ideal point α on the line at infinity, it follows that each of a system of parallel lines contains the same ideal point. And further, no line can contain more than one ideal point, since its corresponding line has only one point of intersection with $lx+my+n=0$. Therefore every class of parallel lines determines a unique (ideal) point, just as every class of concurrent lines determines a unique (finite) point. Consequently the direction of a line or a class of parallel lines may be identified with the ideal point they determine. Now the line at infinity contains ideal points associated with every possible direction. It is therefore meaningless to associate any determinate direction with it: for it is, so to speak, in itself the aggregate of all possible directions.

It may be noted in passing that the equations, on which our analysis is based, tacitly assume a Euclidean axiom of parallelism. The statement that every line contains one and only one ideal point is merely an alternative mode of expression of Playfair's

axiom. If, as in non-Euclidean geometry, this is discarded, it is no longer possible to say that every line contains *either* one *or* only one ideal point. Summarising this discussion, we conclude that the term "line at infinity" is introduced solely to secure continuity of statement in dealing with homographic transformations. It is a line, only in the sense that it corresponds homographically to a (finite) straight line, which involves the fact that it meets every other straight line in one and only one (ideal) point. And it lies at infinity, only in the sense that it contains no finite point: in other words, the statement that it lies at infinity indicates its character but not its position. Lastly, it is devoid of any sense of direction, and is consequently incapable of any exact graphical representation.

An interesting example of the necessity for care in working with imaginary and ideal elements is supplied by Inversion. We propose

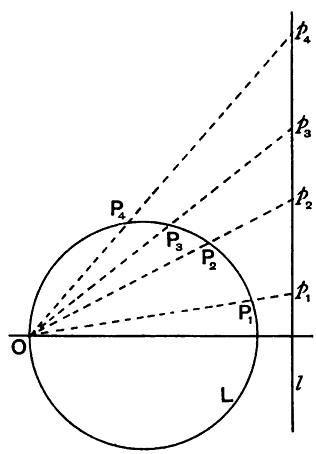

FIG. I.

to consider what the inverse of a circle L is, when the centre of inversion lies on L.

From a geometrical standpoint, if P_1, P_2, P_3, ... are a series of points on L, and if O is the centre of inversion, the inverse points p_1, p_2, p_3, ... lie on a straight line l, parallel to the tangent at O (Part I., p. 135). If P_n is very close to O, then p_n lies on l at a great distance from O; and as P_n tends to O, the line OP_n tends to assume the direction of the tangent at O to L. If, then, we regard inversion as a continuous process, the inverse of P_n, in

the limit when $P_n \to O$, is the point at infinity on l. Consequently it would appear that the inverse of L is the finite line l, together with the ideal point on l.

Suppose now that O is a point very close to L, then the inverse l is a circle, the length of whose radius $\to \infty$, as O tends to L. It would therefore appear that, in the limit when O lies on L, the inverse l degenerates into a finite line together with the line at infinity.

In this way we have two distinct interpretations of what the inverse figure is: this really means that our definition of inversion is not sufficiently precise.

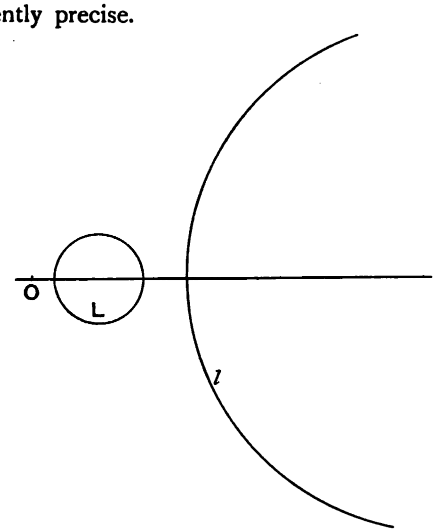

FIG. 2.

Analytically, the equations of the inversion-transformation are easily seen to be

$$\xi = \frac{k^2 x}{x^2 + y^2}, \quad \eta = \frac{k^2 y}{x^2 + y^2}.$$

Provided that $x^2 + y^2 \neq 0$ and $\xi^2 + \eta^2 \neq 0$, these equations can be solved without difficulty for x and y, and give

$$x = \frac{k^2 \xi}{\xi^2 + \eta^2}, \quad y = \frac{k^2 \eta}{\xi^2 + \eta^2}.$$

With these restrictions, the equations always admit of interpretation, and supply a correspondence which is always (1, 1). But if these restrictions are removed, the equations may be meaningless or cease to give a (1, 1) correspondence.

Now the value of the process of inversion depends largely on the fact that it is in general (1, 1). Suppose, for example, it is required to prove that three curves have a common point. If, after inversion, the three corresponding curves are known to have a common point h, it is possible to argue that the original curves had as a common point the inverse H of h, since to the point h corresponds one and only one point H. Should however the correspondence not be (1, 1), this argument would be invalid, because to the point h might then correspond three distinct points H_1, H_2, H_3. It is therefore **convenient** to define the process of inversion as follows:

"The **process of inversion** is equivalent to the transformation $\xi=\frac{k^2x}{x^2+y^2}$, $\eta=\frac{k^2y}{x^2+y^2}$ applied to all points of the x, y plane, excluding the line at infinity and the isotropic lines through the origin."

It follows from this definition that the inverse of a circle, w.r.t. a point on it, is a straight line with the ideal point on the line omitted.

The advantage of this proviso is further illustrated by the following fallacious argument:

Let two circles L_1, L_2 intersect at the centre of inversion O; the point O inverts into a point o at infinity; therefore the inverses l_1, l_2, which we know are straight lines, have a common point o at infinity, and are therefore parallel.

Therefore all circles through the origin invert into parallel lines, which is absurd.

41. Prove that the inversion transformation is given by

$$\xi=\frac{k^2x}{x^2+y^2},\ \eta=\frac{k^2y}{x^2+y^2},$$

where k is the radius of inversion.

42. Consider the effect of the inversion transformation, when the radius of inversion tends to zero.

43. What is the inverse of the pair of isotropic lines, given by $(x-a)^2+(y-b)^2=0$, w.r.t. the origin?

44. If a homographic transformation is defined by

$$\xi=\frac{2x-y+1}{x+y},\ \eta=\frac{x+2y-1}{x+y},$$

determine the line in the ξ, η plane, which corresponds to the line at infinity in the x, y plane.

45. Determine the general homographic transformation, such that the lines corresponding to the lines at infinity are $x=0$ and $\xi=0$.

46. With the notation of the general homographic transformation, find the condition that $\eta=m_1\xi$ and $\eta=m_2\xi$ correspond to parallel lines in the x, y plane.

47. Determine the general homographic transformation, such that isotropic lines through the origin correspond to isotropic lines through the origin, and the line $x+y=1$ to the line at infinity.

48. Find the equation of the conic, corresponding to the circle $x^2+y^2=1$, in the transformation $x=\dfrac{2\xi+\eta}{\xi-1}$, $y=\dfrac{\xi-2\eta}{\xi-1}$.

ANALYTICAL REPRESENTATION OF POINTS AT INFINITY.

Let the coordinates of any point P referred to rectangular axes OX, OY be $\left(\dfrac{x}{z}, \dfrac{y}{z}\right)$, where x, y, z are always connected by the relation $x+y+z=1$.

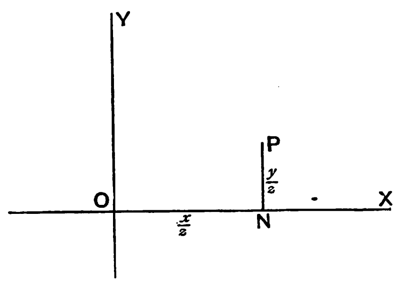

FIG. 3.

To every position of P there corresponds a unique set of values for x, y, z. Any straight line can be represented by an equation of the form $a\left(\dfrac{x}{z}\right)+b\left(\dfrac{y}{z}\right)+c=0$ or $ax+by+cz=0$.

Any two parallel lines can be represented by $y-mx=c_1z$; $y-mx=c_2z$.

To obtain their (ideal) point of intersection we have, by subtraction, $(c_1-c_2)z=0$ or $z=0$.

Hence all ideal points satisfy the condition $z=0$: we therefore express this by saying that $z=0$ may be regarded as the **equation of the line at infinity**, in the plane considered.

With the same notation the equation of any circle can be written in the form $x^2+y^2+2gxz+2fyz+cz^2=0$.

This meets the line $z=0$ where $x^2+y^2=0$ or $y=\pm ix$.

Hence every circle cuts the line at infinity in the same two points, viz. $\frac{x}{1}=\frac{y}{\pm i}=\frac{1}{1\pm i}$; $z=0$; since $x+y=1$.

These two points are called the **circular points at infinity**, and will be denoted by ω, ω'.

49. Explain the sense in which the term "point" is used in speaking of the circular points at infinity.

THEOREM 18.

Two concentric circles have double contact with each other at the circular points at infinity.

Take the origin at their common centre.

Their equations take the form $\begin{cases} x^2+y^2-a^2z^2=0, \\ x^2+y^2-b^2z^2=0. \end{cases}$

Their intersections are therefore given by $\begin{cases} x^2+y^2=0, \\ z^2=0. \end{cases}$

$\therefore$ they touch each other at the points given by $\frac{x}{1}=\frac{y}{\pm i}$, $z=0$; *i.e.* at ω, ω'.

Q.E.D.

THEOREM 19.

If two conics are homographically related to two concentric circles, the two conics must have double contact with each other.

A contact can be regarded as the limit of an intersection at two adjacent points: hence, if two curves touch each other, any two curves, homographically related to them, must also touch each other. The theorem therefore follows from Theorem 18.

Q.E.D.

THEOREM 20.

Every conic which passes through the circular points at infinity must be a circle.

The equation of any conic can be written in the form

$$ax^2+2hxy+by^2+2gxz+2fyz+cz^2=0. \quad \text{[Theorem 17]}$$

This meets $z=0$ where $ax^2+2hxy+by^2=0$.

Therefore, by hypothesis, the points

$$\left.\begin{array}{r} ax^2 + 2hxy + by^2 = 0, \\ z = 0 \end{array}\right\} \text{ and } \left.\begin{array}{r} x^2 + y^2 = 0, \\ z = 0 \end{array}\right\}$$

are identical;

$$\therefore\ a = b \text{ and } h = 0;$$

$$\therefore \text{ the conic is a circle.}$$

Q.E.D.

Although the last few theorems have been stated in geometrical terms, it must be clearly understood that such geometrical language is employed merely as a convenient means of expressing the result of a certain analytical process. And it is only possible to make use of a geometrical notation if it is agreed initially that the terms introduced contain a wider significance than is attached to them in ordinary graphical work. The existence of the circular points at infinity does not correspond to any graphical reality, but merely affords a conventional and suggestive interpretation, by a geometrical channel, of an analytical phenomenon. It expresses a definite feature of the properties of a circle defined analytically, *i.e.* by an equation, viz. the terms of second degree in the equation of every circle are x^2+y^2, and conversely, if an equation is of the second degree, with $x^2+y^2+0 \, . \, xy$ as its leading terms, it must represent a circle. But while it is convenient to generalise our notation, it is at the same time necessary to avoid making other uses of this notation than are warranted by the underlying analytical structure.

The chief analytical properties of circular points and isotropic lines have been already indicated. Their application to pure geometry rests on two facts: (1) that the circular points can be made to correspond homographically to any two given points; (2) the cross ratio of a pencil of four concurrent lines or of a range of four collinear points is unaltered in value by any homographic transformation.

The latter has been already established: we shall now proceed to prove the former.

THEOREM 21.

To determine a simple homographic transformation by which two given points correspond to the circular points at infinity.

Let A, B be the given points. Choose the y-axis so as to bisect AB at right angles.

The coordinates of A, B are (f, b), $(-f, b)$ respectively.

The line $y=b$ is, by hypothesis, to correspond to the line at infinity;

$$\therefore\ \xi=\frac{\alpha_1 x+\beta_1 y+\gamma_1}{y-b},\quad \eta=\frac{\alpha_2 x+\beta_2 y+\gamma_2}{y-b};$$

$$\therefore\ \frac{\xi}{\eta}=\frac{\alpha_1 x+\beta_1 y+\gamma_1}{\alpha_2 x+\beta_2 y+\gamma_2}.$$

Now (f, b) is to correspond to $\frac{\xi}{1}=\frac{\eta}{i}$;

$$\therefore\ \frac{1}{i}=\frac{\alpha_1 f+\beta_1 b+\gamma_1}{\alpha_2 f+\beta_2 b+\gamma_2},\ \text{and similarly}\ -\frac{1}{i}=\frac{-\alpha_1 f+\beta_1 b+\gamma_1}{-\alpha_2 f+\beta_2 b+\gamma_2};$$

$$\therefore\ i\alpha_1 f+i\beta_1 b+i\gamma_1=\alpha_2 f+\beta_2 b+\gamma_2$$

and

$$i\alpha_1 f-i\beta_1 b-i\gamma_1=-\alpha_2 f+\beta_2 b+\gamma_2;$$

$$\therefore\ i\alpha_1 f=\beta_2 b+\gamma_2\ \text{and}\ i\beta_1 b+i\gamma_1=\alpha_2 f.$$

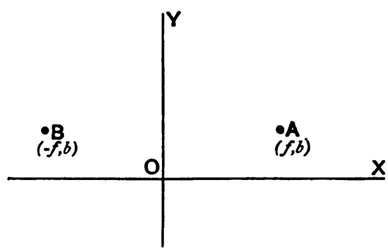

FIG. 4.

We have therefore two equations to satisfy and six independent constants at our disposal.

To obtain as simple a form as possible, we will therefore suppose that β_1, γ_1, α_2, γ_2 are zero: and then choose α_1, β_2, so that

$$i\alpha_1 f=\beta_2 b\ \text{ or }\ \frac{\alpha_1}{b}=\frac{\beta_2}{if}.$$

Hence a simple form of transformation is given by

$$\xi=\frac{bx}{y-b},\quad \eta=\frac{ify}{y-b}.$$

Q.E.D.

50. Prove that the general homographic transformation which makes the two points (0, 0), (h, k), in the (x, y) figure correspond to the circular points in the (ξ, η) figure is given by

$$\xi=\frac{a_1 x+b_1 y+\frac{1}{2}h(ia_2-a_1)+\frac{1}{2}k(ib_2-b_1)}{kx-hy},$$

$$\eta=\frac{a_2 x+b_2 y-\frac{1}{2}h(a_2+ia_1)-\frac{1}{2}k(b_2+ib_1)}{kx-hy}.$$

51. Prove that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is transformed into a circle by $x=a\xi$, $y=b\eta$.

52. Prove that the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is transformed into a circle by $x=a\xi$, $y=ib\eta$.

53. Determine a simple transformation which will make the points $(a, 2a)$; $(a, -2a)$ correspond to the circular points: and verify that this transformation changes the parabola $y^2=4ax$ into a circle.

54. Determine a simple transformation which will make the points $\left(\frac{a}{2}, \frac{b\sqrt{3}}{2}\right)$; $\left(-\frac{a}{2}, \frac{b\sqrt{3}}{2}\right)$ correspond to the circular points; verify that it transforms the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ into a circle.

55. Prove that a conic, given by its general equation, can in general be transformed into a circle by relations of the form

$$x=\frac{a}{\eta+\nu}, \quad y=\frac{\beta\eta}{\eta+\nu}.$$

56. Determine a simple homographic transformation, which makes $x^2+3xy+4y^2+1=0$ correspond to a circle. [Use Ex. 55.]

57. Determine a simple homographic relation, transforming the conics $x^2+3y^2=1$ and $3x^2+y^2=2$ into circles.

58. $S_1=0$, $S_2=0$ are the equations of two conics; prove that any transformation which makes them correspond to circles, will also make the conic $S_1-\lambda S_2=0$ correspond to a circle, where λ is any constant.

This completes our summary of the rudimentary principles, governing the part played by imaginary elements in pure geometry, viewed from an analytical source. This is the historical path by which such ideas first penetrated into geometrical reasoning: and it is generally true that the order of historical discovery provides the most suggestive and intelligible course for the student.

Since the process of conical projection is homographic, and since the constants which appear in the equations of transformation may be real or complex, the foregoing pages supply a complete justification for the validity of imaginary conical projection, *i.e.* projection where either the vertex or any points or lines in the projected figure may be imaginary. The process is to be regarded as analytical (see p. 76). The statement of the mode of projection in geometrical language is only a convenient means of indicating the way in which the homographic relation is to be chosen.

This analytical treatment has been preferred as being more simple, and at the same time no less rigorous, than a purely geometrical method. It is, however, possible to build up a consistent logical theory of imaginary elements without any reference to analysis at all. This is effected, for example, by Von Staudt in his great work, *The Geometry of Position*, which is concerned solely with the disposition of systems of points, lines, planes, and curves generated from them. He builds up a non-metrical theory of harmonic, homographic and involution systems; and then evolves the conception of imaginary elements, in conjugate pairs, to secure continuity of statement, in connection with the theory of double points and lines.

CHAPTER II.

ORTHOGONAL PROJECTION.

THIS chapter deals with the simplest method of generating homographically one plane figure from a given plane figure.

Imagine any geometrical system, composed of points, lines and curves, drawn on a flat glass plate. If this is held up in the sunlight above a sheet of paper, shadows will be cast on the paper, forming a second geometrical system, corresponding in every detail to the first. If the sun is directly overhead and if the sheet of paper is placed in a horizontal position, the light-rays are all perpendicular to the plane of the paper. In this case the shadow system on the paper is called the **orthogonal projection** of the geometrical system on the glass plate.

The correspondence between the two systems is complete, in the sense that every point, line, or curve in the one is directly connected with a unique point, line, or curve in the other. But it is clear that in general the shapes and sizes of the two figures will differ. The shadow of any object gives only a distorted representation of that object. For example, it will be shown that the shadows of two equal lines are lines of equal length, if and only if the lines are parallel: provided that the valueless case, in which the planes of the glass and paper are parallel, is excluded.

Definitions.

(1) P_1, P_2, ... are a system of points in a plane Σ; p_1, p_2, ... are the feet of the perpendiculars from these points to a plane σ.

Then the system p_1, p_2, ... is called the **orthogonal projection** of the system P_1, P_2, ... on the plane σ.

(2) The line of intersection of the planes Σ and σ is called the **axis of projection**.

We proceed to enumerate a number of simple theorems on which the utility of orthogonal projection depends. Many still hold good in the more general process of conical projection: in such cases, the enunciation is marked with an asterisk.

In the present chapter, the word "projection" is used as an abbreviation of "orthogonal projection."

Unless otherwise stated, capital letters will be used to refer to the original system and small letters to the corresponding elements of the projected system.

*THEOREM 22.

(1) A straight line projects into a straight line.

(2) The meet of two straight lines or curves projects into the meet of their projections.

(3) The join of two points projects into the join of their projections.

(4) Any point on the axis of projection is unaltered in position by projection.

(5) Any line and its projection meet on the axis of projection.

The proof is left to the reader.

THEOREM 23.

Parallel lines project into parallel lines.

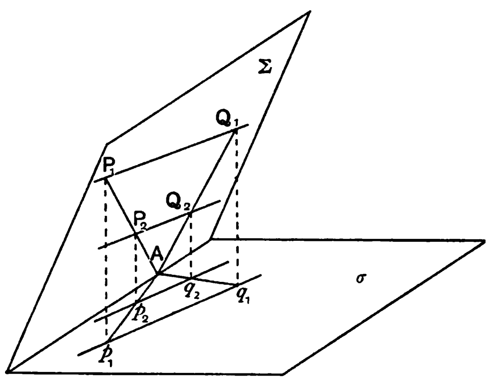

FIG. 5.

Take any point A on the axis of projection, and draw any two lines AP_2P_1, AQ_2Q_1 to cut the parallel lines at P_2, P_1 and Q_2, Q_1.

Let p_2, p_1, q_2, q_1 denote their projections on σ.

Then, by parallels, $\dfrac{Ap_1}{Ap_2}=\dfrac{AP_1}{AP_2}=\dfrac{AQ_1}{AQ_2}=\dfrac{Aq_1}{Aq_2}$;

$\therefore$ p_1q_1 is parallel to p_2q_2. Q.E.D.

Now we saw in Chapter I. that two or more parallel lines determine an ideal point, called a point at infinity, and that the aggregate of ideal points, determined by taking all possible systems of parallel lines, constitutes an ideal line, called the line at infinity. Since, then, parallel lines project into parallel lines, it follows that an ideal point of Σ projects into an ideal point of σ, and consequently the line at infinity in Σ projects into the line at infinity in σ. This fact may at first sight appear to need no proof, since it might be argued that a line, parallel for example to the axis of projection and at a great distance from it, projects into a line also situated at a great distance from the axis of projection: and that consequently a line "infinitely distant" must project into a line "infinitely distant."

To see the error in this argument, it is only necessary to refer to Chapter I., where it was pointed out that the phrase "infinitely distant" has no metrical meaning, and that the "line at infinity" has no geographical quality. If any logical use is to be made of the notion of "infinity," it is essential that the properties associated with it should be deduced rigorously from the initial definitions.

THEOREM 24.

(1) If P_2 is the mid-point of P_1P_3, then p_2 is the mid-point of p_1p_3.

(2) If P_1, P_2, P_3 are collinear, then $\dfrac{P_1P_2}{P_2P_3}=\dfrac{p_1p_2}{p_2p_3}$.

*(3) If P_1, P_2, P_3, P_4 are collinear, and if $\{P_1P_2P_3P_4\} = -1$, then $\{p_1p_2p_3p_4\} = -1$.

*(4) If P_1, P_2, P_3, P_4 are collinear, then the ranges $\{P_1P_2P_3P_4\}$, $\{p_1p_2p_3p_4\}$ are equicross.

The proof is left to the reader.

THEOREM 25.

If P_1Q_1, P_2Q_2 are parallel, then $\dfrac{P_1Q_1}{P_2Q_2}=\dfrac{p_1q_1}{p_2q_2}$.

The proof is left to the reader.

[Let P_1P_2 meet Q_1Q_2 at A; take the projection a of A, and use the method of Theorem 23.]

THEOREM 26.

(1) If P_1P_2 is parallel to the axis of projection, then $P_1P_2 = p_1p_2$.

(2) If P_1P_2 is perpendicular to the axis of projection, and if θ is the angle of intersection of the two planes, then

$$p_1p_2 = P_1P_2 \cdot \cos\theta.$$

The proof is left to the reader.

1. Prove Theorem 22.

2. Prove Theorem 24.

3. Prove Theorem 25.

4. Prove Theorem 26.

5. If G is the centroid of the triangle ABC, prove that g is the centroid of the triangle abc.

6. ABC is an equilateral triangle: if BC is parallel to the axis of projection and if the two planes are inclined at an angle of 45°, calculate the angle *abc*.

7. AB makes an angle α with the axis of projection: if θ is the angle of inclination of the two planes, prove that *ab* makes an angle $\tan^{-1}(\tan\alpha \cdot \cos\theta)$ with the axis of projection.

8. If AB equals 4 cms. and makes with the axis of projection an angle of 30°, find the length of its projection on a plane making an angle of 60° with the first plane.

9. If the axis of projection is taken as the x-axis and if the coordinates of A, B are $(2, 3)$, $(-1, -4)$, find the length of *ab*, taking 45° as the angle of inclination of the two planes.

10. With the data of Ex. 9, determine the projection of $2x^2+y^2=r^2$.

11. Taking the axis of projection as x-axis, find the angle of inclination of the planes in order that $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may project into a circle, given that $a<b$.

12. AB, BC are respectively perpendicular and parallel to the axis of projection; if θ is the angle of inclination of the two planes, prove that $\frac{\triangle abc}{\triangle ABC}=\cos\theta$.

Hence if ABC is any triangle, prove that $\frac{\triangle abc}{\triangle ABC}=\cos\theta$.

[Circumscribe about ABC a rectangle having one side parallel to the axis of projection, and one corner at a vertex of the triangle.]

13. If A is the area of any plane rectilinear figure, and if a is the area of its projection on a plane making an angle θ with the first plane, prove that $a = A \cdot \cos\theta$.

14. Prove that the mean centre of any system of coplanar points projects into the mean centre of their projections.

15. AB, CD are two equal perpendicular lines of constant length; prove that ab^2+cd^2 is constant, for a given plane of projection.

16. Prove that the degree of a curve is unaltered by projection.

17. ABC is an equilateral triangle of constant size; prove that $ab^2+bc^2+ca^2$ is constant.

18. If the sum of the projections of any number of given lines is equal to the projection of another given line, on each of two lines, prove that the same is true, if the projections are taken on any other line, all the lines being coplanar.

ANALYTICAL TREATMENT.

The connection between a figure and its projection can be expressed very simply in analytical terms.

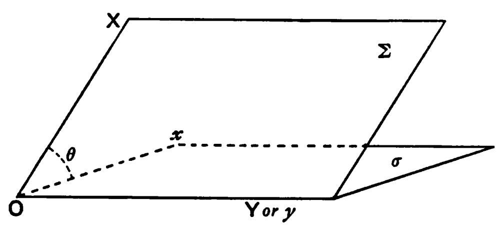

FIG. 6.

Choose the axis of projection as the Y-axis in Σ and the y-axis in σ. From any point O on the axis of projection, draw OX, Ox perpendicular to the axis of projection, in the planes Σ, σ.

Let (X, Y) be the coordinates of any point P in Σ, and let (x, y) be the coordinates of its projection p on σ.

Let θ be the angle of inclination of Σ and σ.

Then by Theorem 26, $\begin{cases} x = X\cos\theta, \\ y = Y. \end{cases}$

These two equations form the homographic transformation connecting a figure and its orthogonal projection.

THEOREM 27.

(1) If the side AB of the rectangle $ABCD$ is parallel to the axis of projection, then the ratio of the areas $abcd$, $ABCD$ is equal to $\cos\theta$, where θ is the angle of inclination of the two planes.

(2) If S is the area of any closed figure in Σ, and if s is the area of its projection on σ, then $s = S . \cos\theta$.

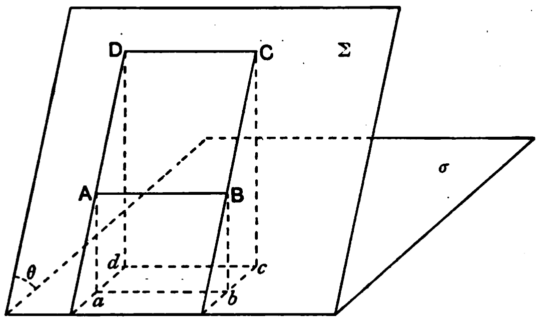

FIG. 7.

(1) By Theorem 26, $ab = AB$ and $bc = BC . \cos\theta$;

$$\therefore\ ab . bc = AB . BC . \cos\theta.$$

But bc is perpendicular to the axis of projection, and therefore to ab.

Therefore the ratio of the areas $abcd$, $ABCD$ is equal to $\cos\theta$.

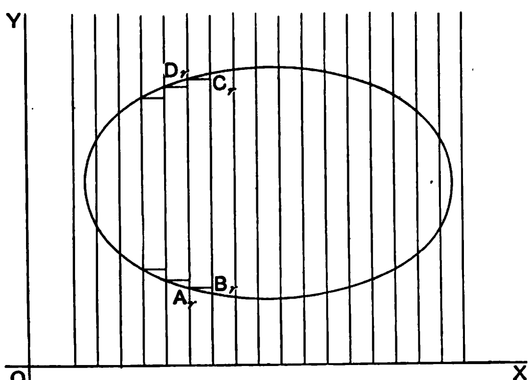

FIG. 8.

(2) Draw a number of lines, distant h apart, perpendicular to the axis of projection, so that the closed curve is divided into a number of strips, each of breadth h: in each strip a rectangle is inscribed, of breadth h, having the shorter straight edge of the strip as one of its sides.

By taking h sufficiently small, it can be proved* that the sum

* This was done by Newton in the opening lemmas of the *Principia*.

of the areas of all these rectangles can be made to differ from the area of the closed curve by less than any assigned quantity, however small. [See Ex. 19.]

If $A_rB_rC_rD_r$ is any one such rectangle, we have, by the first part, $a_rb_rc_rd_r = \cos\theta \,.\, A_rB_rC_rD_r$;

$\therefore$ adding up, for all the rectangles,

$$\Sigma(a_rb_rc_rd_r) = \cos\theta \,.\, \Sigma(A_rB_rC_rD_r);$$

$$\therefore \text{ when } h \to 0,\ s = \cos\theta \,.\, S. \qquad \text{Q.E.D.}$$

Making use of the analytical notation explained above, this theorem may be readily proved by the aid of the integral calculus.

$$s = \int x\,dy = \int X\cos\theta\, dY = \cos\theta \int X\,dY = \cos\theta \,.\, S. \qquad \text{Q.E.D.}$$

19. Inscribe in each strip a rectangle of breadth h, having the longer straight edge of the strip as one side: prove that the area of the curve lies between the sums of the areas of these two systems of rectangles, and that these two sums differ, for any oval curve, by less than $2hd$ where d is the length of the maximum chord of the curve, perpendicular to the axis of projection. Hence establish Newton's result.

THEOREM 28.

(1) If G is the centroid of masses $m_1, m_2, \ldots$ at $P_1, P_2, \ldots$, then g is the centroid of masses $m_1, m_2, \ldots$ at $p_1, p_2, \ldots$.

(2) If G is the centroid of an area S, then g is the centroid of the area s.

The notation is as above.

(1) Let $(\bar{X}, \bar{Y})$; (X_1, Y_1); ... be the coordinates of G, P_1 ....

Then $\bar{X} = \dfrac{\Sigma(mX)}{\Sigma m}$; but $\bar{X} = \bar{x}\sec\theta$; $X_1 = x_1\sec\theta$; ... ;

$$\therefore\ \bar{x}\sec\theta = \frac{\Sigma(mx\sec\theta)}{\Sigma m} = \sec\theta\,\frac{\Sigma(mx)}{\Sigma m}.$$

$$\therefore\ \bar{x} = \frac{\Sigma(mx)}{\Sigma m}.$$

Also $$\bar{Y} = \frac{\Sigma(mY)}{\Sigma m};\ \text{ but } \bar{Y} = \bar{y};\ \ Y_1 = y_1; \ldots$$

$$\therefore\ \bar{y} = \frac{\Sigma(my)}{\Sigma m};$$

$\therefore$ $(\bar{x}, \bar{y})$ or g is the centroid of $m_1, m_2, \ldots$ at $p_1, p_2 \ldots$.

Q.E.D.

(2) Divide the area S into a system of rectangles, as in Theorem 27.

The centroid of each rectangle projects into the centroid of the projection of that rectangle.

Let M_1, M_2, ... be the masses of the rectangles which make up S, and m_1, m_2, ... the masses of the corresponding rectangles in s; then $\dfrac{m_1}{M_1}=\dfrac{m_2}{M_2}=\ldots=\cos\theta$, by Theorem 27 (1).

Now, when $h \to 0$, G is the centroid of masses M_1, M_2, ... at the centres of the rectangles of S, therefore its projection g is the centroid of masses M_1, M_2, ... at the centres of the rectangles of s, and is therefore the centroid of the proportional masses m_1, m_2, ... at these points, or in other words, is the centroid of s.

Q.E.D.

With the notation of the integral calculus, this proof may be written as follows:

$$x\text{-coordinate of centroid of } s=\frac{\iint \rho x dx dy}{\iint \rho dx dy}$$

$$=\frac{\iint \rho \cos^2\theta X dX dY}{\iint \rho \cos\theta\, dX dY}=\cos\theta\frac{\iint \rho X dX dY}{\iint \rho dX dY}$$

$$=\cos\theta\,.\,\overline{X},$$

and similarly y-coordinate of centroid of $s=\overline{Y}$. Q.E.D.

20. $ABCD$ is a square; AB makes an angle of 30° with the axis of projection: the planes are inclined at an angle of 60°, find the value of $\sin d\hat{a}b$.

21. $OABC$ is a tetrahedron; $OA=OB=OC=x$; $AB=BC=CA=y$; find the angle of intersection of the planes OBC, ABC.

22. $OABC$ is a tetrahedron:

$$A\hat{O}B=B\hat{O}C=C\hat{O}A=90^\circ;\ OA=x,\ OB=y,\ OC=z;$$

find the angle of intersection of the planes OBC, ABC.

23. OA, OB, OC are three mutually perpendicular lines; s_1, s_2, s_3 are the areas of the projections of the triangle ABC on the planes OBC, OCA, OAB; if s is the area of the triangle ABC, prove that $s^2=s_1^2+s_2^2+s_3^2$.

THE ELLIPSE.

The process of orthogonal projection may be applied with considerable success to the geometry of the ellipse. It will appear that many properties of the circle can be made to yield corresponding generalised properties of the ellipse.

To exhibit this connection, we shall define the ellipse as a curve whose equation can be written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

It is evident, by tracing the graph of this equation, that an ellipse is a curve of the shape shown in Fig. 9, symmetrical about two lines ACA', BCB', the x-axis and the y-axis, when the equation is written in its standard form.

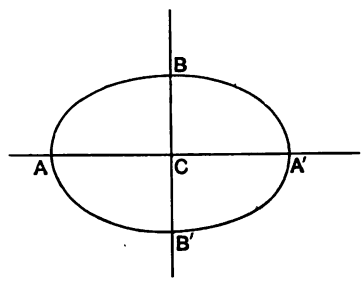

FIG. 9.

If $a>b$, the line $ACA'(=2a)$ is called the **major axis**, and the line $BCB'(=2b)$ is called the **minor axis**, and the point C is called the **centre** of the ellipse. In order to show that an ellipse, as defined above, is a conic, as defined in Chapter IV., it is necessary to prove that any ellipse can be regarded as the conical projection of a circle. This is done in Theorem 29, the vertex of projection being at infinity.

THEOREM 29.

By taking the axis of projection along the minor axis OY of the ellipse $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, $(a>b)$, and by taking for θ the value given by $\cos\theta=\frac{b}{a}$, the ellipse is projected into the circle $x^2+y^2=b^2$.

With the previous notation, $X=\dfrac{x}{\cos\theta}=\dfrac{ax}{b}$; $Y=y$;

$$\therefore\ \frac{X^2}{a^2}+\frac{Y^2}{b^2}=1 \text{ becomes } \frac{1}{a^2}\left(\frac{ax}{b}\right)^2+\frac{y^2}{b^2}=1,$$

or

$$x^2+y^2=b^2. \qquad \text{Q.E.D.}$$

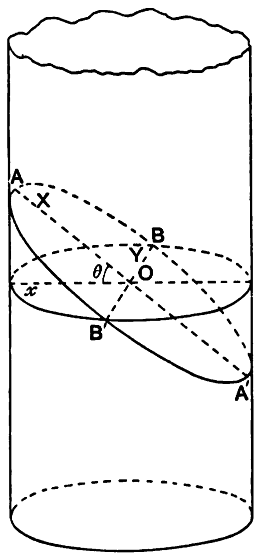

FIG. 10.

THEOREM 30.

Ellipses, which have their major axes parallel, and the ratio of their major to their minor axis constant, can be projected into a system of circles.

The equation of any ellipse of the system can be written:

$$\frac{(X-g_r)^2}{r^2a^2}+\frac{(Y-f_r)^2}{r^2b^2}=1,$$

where g_r, f_r, r vary.

Project as in Theorem 29, then the equation becomes

$$\frac{\left(\frac{ax}{b}-g_r\right)^2}{r^2a^2}+\frac{(y-f_r)^2}{r^2b^2}=1,$$

or

$$\left(x-\frac{bg_r}{a}\right)^2+(y-f_r)^2=r^2b^2,$$

which is a circle. Q.E.D.

Definition.

Ellipses which have their major axes parallel, and the ratio of their major to their minor axis constant, are said to be similar and similarly situated or **homothetic.**

Hence Theorem 30 can be stated in the form:

A system of homothetic ellipses can be projected into a system of circles.

And it is easy to see from the work of Theorem 30 that:

"A system of concentric homothetic ellipses can be projected into a system of concentric circles."

It should be noted that if a circle is the orthogonal projection of an ellipse, the axis of projection is parallel to the minor axis, since the semi-major axis has to be contracted from a to b; but if an ellipse is the orthogonal projection of a circle, then the axis of projection is parallel to the major axis. The reader who is acquainted with the geometry of the ellipse will see that the connection between a circle and its orthogonal projection is identical with the connection between the auxiliary circle of an ellipse and the ellipse itself. [See Ex. 36.]

24. Find θ in order that $y^2=4ax$ may project into $y^2=4bx$.

25. Find θ in order that the circle $x^2+y^2=r^2$ may be projected into the ellipse $\frac{x^2}{r^2}+\frac{r^2+1}{r^2}y^2=1$.

26. Prove that concentric homothetic ellipses can be projected into concentric circles.

27. Prove that any system of circles project into homothetic ellipses.

28. Prove that any hyperbola can be projected into a rectangular hyperbola.

29. Prove that the projection of a parabola is a parabola.

30. If the circle $\mathbf{X^2+Y^2=a^2}$ is projected into the ellipse $\mathbf{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$, prove that the tangent at $(\mathbf{a}\cos\phi, \mathbf{a}\sin\phi)$ on the circle projects into the tangent at $(\mathbf{a}\cos\phi, \mathbf{b}\sin\phi)$ on the ellipse: and that the eccentric angle of any point is unaltered by projection.

31. With the notation of Ex. 30, prove that the polar of (X, Y) w.r.t. the circle projects into the polar of (x, y), the projection of (X, Y), w.r.t. the ellipse, assuming the ordinary analytical formula for the polar of a point.

32. Assuming the theorem: "the mid-points of a system of parallel chords of a circle lie on a straight line," establish, by projecting an ellipse into a circle, the corresponding property of the ellipse.

33. Prove by projection that there exists one point C in the plane of an ellipse such that every chord through C is bisected at C.

34. With the notation of Ex. 33, if PCQ is any chord through C, prove that the tangents at P, Q are parallel.

35. With the notation of Ex. 33, if the tangents at the extremities of a chord HK meet at T, prove that CT bisects HK.

36. With the notation on page 38, if N is the foot of the perpendicular from a variable point P of an ellipse, semi-axes a, b, to its major axis AA′, and if NP is produced to Q so that $\frac{NP}{NQ}=\frac{b}{a}$, prove that the locus of Q is a circle (called the auxiliary circle), having AA′ as a diameter.

37. A variable line PQR cuts two fixed perpendicular lines CA, CB at Q, R; if PQ=b, PR=a, where a, b are constants, prove that P traces out an ellipse, having its semi-axes equal to a, b and situated along CA, CB. [This theorem gives a mechanical method for describing an ellipse.]

*THEOREM 31.

A tangent to a curve projects into a tangent to the projected curve.

The tangent at any point P of a curve is the limiting position of the chord PQ as $Q \to P$. The theorem is therefore self-evident.

Definition.

If a variable line is drawn through a fixed point P to cut an ellipse at H, K, and if Q is the harmonic conjugate of P w.r.t. H, K, then the locus L of Q is called the **polar** of P w.r.t. the ellipse, and Q is called the **pole** of L.

THEOREM 32.

(1) The polar of P w.r.t. an ellipse is a straight line.

*(2) A point and its polar w.r.t. an ellipse project into a point and its polar w.r.t. the circle, into which the ellipse is projected.

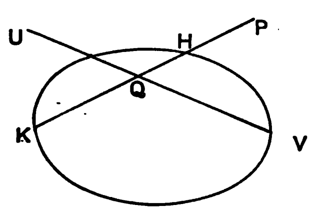

FIG. 11.

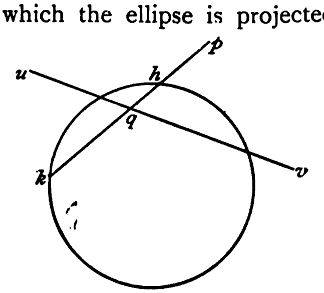

FIG. 12.

Let any line through P cut the ellipse at H, K; let Q be the harmonic conjugate of P w.r.t. H, K. Project the ellipse into a circle.

Then by Theorem 24 (3), $\{pq\,;\,hk\}$ is harmonic and p is a fixed point, therefore the locus of q is a straight line uv, which is the polar of p w.r.t. the circle.

$\therefore$ the locus of Q is a straight line UV.

$\therefore$ the polar of P w.r.t. the ellipse is a straight line UV; and the projection of a point and its polar is a point and its polar.

Q.E.D.

THEOREM 33.

(1) If the polar of P w.r.t. an ellipse passes through Q, then the polar of Q passes through P. [P, Q are called **conjugate points** w.r.t. the ellipse.]

(2) If the pole of the line HK w.r.t. an ellipse lies on the line MN, then the pole of MN lies on HK. [HK, MN are called **conjugate lines** w.r.t. the ellipse.]

(3) Conjugate points and lines w.r.t. an ellipse project into conjugate points and lines w.r.t. the circle into which the ellipse is projected.

The proof is left to the reader.

THEOREM 34.

(1) There exists a point inside any ellipse, called its **centre**, which is such that any chord of the ellipse through it, called a **diameter**, is bisected at that point.

(2) If an ellipse is projected into a circle, its centre projects into the centre of the circle.

(3) The locus of the mid-points of a system of parallel chords of an ellipse is a diameter of the ellipse.

(4) PCP', QCQ' are two diameters of an ellipse; if PCP' bisects chords parallel to QCQ', then QCQ' bisects chords parallel to PCP': and PCP', QCQ' are called **conjugate diameters** of the ellipse.

(5) If an ellipse is projected into a circle, any pair of conjugate diameters project into a pair of diameters at right angles of the circle.

The proof is left to the reader.

[Project the ellipse into a circle and use Theorems 23, 24.]

38. Prove Theorem 33.

39. Prove Theorem 34.

40. Obtain by projection a theorem for two ellipses from the following: a tangent at a point O on a circle cuts a concentric circle at P, Q, then $PO = OQ$.

41. Obtain by projection a theorem from the following: T is the pole of a chord PQ of a circle, centre O, then the four points $TPOQ$ lie on a circle.

42. PQ is a variable chord of an ellipse, centre C; if the triangle CPQ is of constant area, prove that PQ touches a fixed concentric homothetic ellipse.

43. The centroid of the triangle PQR, inscribed in an ellipse, is at the centre of the ellipse; prove that if the ellipse is projected into a circle, the triangle PQR is projected into an equilateral triangle.

44. If CP, CD are conjugate semi-diameters of an ellipse, the tangent at P is parallel to CD.

45. Prove that the centre of an ellipse is the pole of the line at infinity w.r.t. the ellipse.

46. Prove that conjugate diameters are conjugate lines w.r.t. the ellipse (*i.e.* either diameter contains the pole of the other).

47. If T is a point outside an ellipse, prove that the chord of contact of the tangents from T is the polar of T.

48. $ABCD$ is a quadrangle inscribed in an ellipse; EFG is its diagonal point triangle; prove that EFG is a self-conjugate triangle w.r.t. the ellipse. [*i.e.* each vertex is the pole of the opposite side.]

49. T is the pole of the chord PQ of an ellipse, centre C; if CT meets PQ at V, prove that $PV = VQ$.

50. With the notation of the last exercise, if CT meets the ellipse at H, prove that $\mathbf{CV \cdot CT = CH^2}$.

51. Given a ruler only, construct the polar of a given point w.r.t. a given ellipse.

52. Generalise the harmonic properties of the quadrilateral circumscribing a circle so as to obtain corresponding theorems for the ellipse.

53. AB, CD are chords of an ellipse, intersecting at O; the tangents at A, D meet at P; the tangents at B, C meet at Q; prove that P, O, Q are collinear.

54. Prove that the cross ratio of the range formed by four collinear points is equal to the cross ratio of the pencil formed by their four polars w.r.t. an ellipse.

In order to illustrate the application of orthogonal projection, the following examples are added. The reader should note that lengths of lines are altered by projection. Consequently, in order to prove metrical properties, it is necessary to cast them into the form of ratios of the same or parallel lines, which by Theorem 25 are not affected by projection. This is illustrated by the first example.

EXAMPLE I.

CP, *CD* are two conjugate semi-diameters of an ellipse; two other conjugate semi-diameters meet the tangent at *P* in *H*, *K*; then $HP \cdot PK = CD^2$.

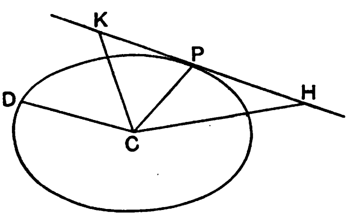

FIG. 13.

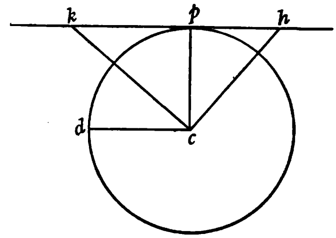

FIG. 14.

Project the ellipse into a circle.

Since *cp*, *cd* are conjugate diameters of the circle, $\widehat{pcd} = 90°$; therefore *cd* is parallel to *hk*; and therefore *CD* is parallel to *HK*.

$\therefore$ each of the ratios $\dfrac{HP}{CD}, \dfrac{PK}{CD}$ is unaltered by projection.

Since *ch*, *ck* are conjugate diameters of the circle, $\widehat{hck} = 90°$;

$$\therefore\ hp \cdot pk = cp^2 = cd^2,$$

$$\therefore\ \frac{hp}{cd} \cdot \frac{pk}{cd} = 1,$$

$$\therefore\ \frac{HP}{CD} \cdot \frac{PK}{CD} = 1, \text{ by Theorem 25,}$$

$$\therefore\ HP \cdot PK = CD^2. \qquad \text{Q.E.D.}$$

EXAMPLE II.

If a parallelogram circumscribes a given ellipse, and if its sides are parallel to a pair of conjugate diameters, then its area is constant.

Project the ellipse into a circle, radius r say; since conjugate diameters project into diameters at right angles, the circumscribing

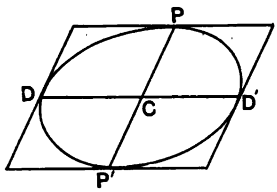

FIG. 15.

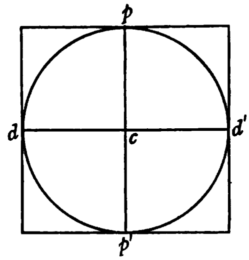

FIG. 16.

parallelogram, area Δ say, becomes a square circumscribing the circle.

The area of the square is therefore $4r^2$.

$$\therefore \text{ by Theorem 27, } \Delta . \cos\theta = 4r^2,$$

where θ is the angle of inclination of the two planes.

$$\therefore \Delta \text{ is constant.}$$ Q.E.D.

EXAMPLE III.

Generalise by orthogonal projection, the following:

Of all triangles that can be inscribed in a given circle, the equilateral triangle has a maximum area: an unlimited number of such triangles exist, and their sides touch a concentric circle.

Project the circle into an ellipse. Now the eccentric angles of the vertices of an equilateral triangle inscribed in a circle differ by $\frac{2\pi}{3}$; but the eccentric angle of a point is unaltered by projection. (see Ex. 30); therefore the eccentric angles of the vertices of a triangle of maximum area, that can be inscribed in an ellipse, are $\phi,\ \phi+\frac{2\pi}{3},\ \phi+\frac{4\pi}{3}$.

Hence we have the theorem:

Of all triangles that can be inscribed in an ellipse, the triangle whose vertices have, as eccentric angles, $\phi,\ \phi+\frac{2\pi}{3},\ \phi+\frac{4\pi}{3}$, is of maximum area: an unlimited number of such triangles exist, and their sides touch a concentric homothetic ellipse.

Another mode of enunciating this theorem is suggested by the property of Ex. 57.

55. Calculate the area of the maximum triangle that can be inscribed in an ellipse, semi-axes a and b.

56. Prove that the centroid of a triangle of maximum area that can be inscribed in an ellipse lies at the centre of the ellipse.

57. If PQR is a triangle of maximum area that can be inscribed in an ellipse, prove that the tangents at P, Q, R are parallel to QR, RP, PQ.

58. P is any point on an ellipse, centre C, major axis ACA'; a line AQ parallel to CP cuts the conic at Q and the minor axis at R, prove that $AQ . AR = 2CP^2$.

59. POQ is a variable chord of an ellipse; O is a fixed point; R is a point on PQ such that $OR^2 = PO . OQ$; find the locus of R.

60. From a fixed point O is drawn a variable line cutting an ellipse at P, Q; CD is a semi-diameter of the ellipse, parallel to OP; prove that $\dfrac{OP . OQ}{CD^2}$ is constant.

61. The sides BC, CA, AB of a triangle touch an ellipse at P, Q, R; prove that BP . CQ . AR = PC . QA . RB.

62. The tangents TP, TQ at the points P, Q on an ellipse are at right angles; PH, QK are the normal chords at P, Q; prove that $TP . PH = TQ . QK$.

63. The tangents at the extremities of a chord PQ of an ellipse meet at T; if the eccentric angles of P, Q differ by a constant, find the locus of T.

64. Prove that the area of the minimum triangle which can be described about an ellipse, semi-axes a, b, is $3\sqrt{3}ab$.

65. The centroid of a triangle PQR inscribed in an ellipse is the centre C of the ellipse; PC meets QR at L and the ellipse at M; prove that $CL = LM$.

66. [Carnot's theorem]. An ellipse meets the sides BC, CA, AB of a triangle at $\mathbf{P_1}$, $\mathbf{P_2}$; $\mathbf{Q_1}$, $\mathbf{Q_2}$; $\mathbf{R_1}$, $\mathbf{R_2}$; prove that

$$\mathbf{BP_1 . BP_2 . CQ_1 . CQ_2 . AR_1 . AR_2 = CP_1 . CP_2 . AQ_1 . AQ_2 . BR_1 . BR_2}.$$

67. From a fixed point on a given ellipse, any two chords are drawn, and through their extremities two other chords are drawn parallel to the first two chords, cutting the ellipse again at P, Q; prove that PQ is fixed in direction.

68. T is a variable point on a tangent to a given ellipse at a fixed point P; from the mid-point M of TP the other tangent MQ is drawn to the ellipse; prove that TQ passes through a fixed point.

69. An ellipse is inscribed in a triangle so as to touch one side at its mid-point; prove that the locus of its centre is a median of the triangle.

70. PP' is a diameter of an ellipse; any chord $P'D$ cuts the tangent at P in Q; prove that the tangent at D bisects PQ.

71. A line cuts two concentric homothetic ellipses at P, Q; H, K; prove that $PH = QK$.

72. H, K are two fixed points on an ellipse; HP, KQ are two variable parallel chords; find the envelope of PQ.

73. A triangle is circumscribed to an ellipse and inscribed in a second concentric homothetic ellipse, whose linear dimensions are twice those of the first ellipse; prove that an unlimited number of such triangles exist, and that the points of contact bisect the sides.

74. The external common tangents of two homothetic ellipses meet at T; a line $TPQQ'P'$ meets the first at P, Q and the second at P', Q'; prove that $TP \, . \, TP' = TQ \, . \, TQ'$.

75. From a point on an ellipse, tangents are drawn to a concentric homothetic ellipse, touching it at P, Q, and meeting the first ellipse at R, S; prove that $PQ = \frac{1}{2}RS$.

76. Prove that the polars of a given point w.r.t. a system of concentric homothetic ellipses are parallel.

77. A tangent to an ellipse meets two conjugate diameters at T, T'; prove that the other tangents from T, T' to the ellipse are parallel.

78. PCP', DCD' are two conjugate diameters of an ellipse, semi-axes CA, CB; prove that AP, AP' are parallel to BD, BD'.

79. Find the locus of the pole of a chord of an ellipse which cuts off a segment of constant area.

80. T is the pole of a chord PQ of an ellipse; if the centroid of the triangle TPQ lies on the curve, find the locus of T.

81. N is the foot of the perpendicular from a point P on an ellipse, centre C, to its major axis AA'; NQ is drawn parallel to AP to meet CP at Q; prove that AQ is parallel to the tangent at P.

82. CP, CD are a variable pair of conjugate semi-diameters of an ellipse. T is a point on the tangent at P such that $\dfrac{TP}{CD}$ is constant; find the locus of T.

83. CP, CD are a variable pair of conjugate semi-diameters of an ellipse; find the locus of the mid-point of PD; and prove that the sector PCD is of constant area.

84. The polars of a variable point P w.r.t. two homothetic ellipses meet at Q; prove that the locus of the mid-point of PQ is a straight line.

85. CP, CP'; CQ, CQ' are two pairs of conjugate semi-diameters of an ellipse; prove that the triangles CPQ, $CP'Q'$ are equal in area.

86. Lines are drawn through the vertices of a triangle inscribed in an ellipse, parallel to the diameters bisecting the opposite sides; prove that these lines are concurrent.

87. A chord PQ of an ellipse S_1, touches a concentric homothetic ellipse S_2 at R; PL, QM, RN are three parallel lines meeting S_1 at L, M, and S_2 at N respectively; prove that $2RN=PL \pm QM$.

88. ABC is a triangle inscribed in an ellipse; from any point P on the curve PL, PM, PN are drawn parallel to the diameters conjugate to BC, CA, AB to meet these lines at L, M, N; prove that L, M, N are collinear.

89. CP, CD are two conjugate semi-diameters of an ellipse; G is the centroid of the sector PCD; GN is drawn parallel to CD to meet CP at N; prove that $\frac{GN}{CD}=\frac{CN}{CP}=\frac{4}{3\pi}$. [If O is the centre and AC a chord of a circle, and if G is the centroid of the sector AOC, then $\frac{OG}{OA}=\frac{2}{3}\frac{\text{chord} AC}{\text{arc} AC}$.]

90. PCP' is any sector of an ellipse, centre C; semi-axes a, b; (x, y) are the coordinates of the centroid of the sector, referred to CA, CB; ϕ, ϕ' are the eccentric angles of P, P'. Prove that

$$\frac{x}{a}=\frac{2}{3}\frac{\sin\phi'-\sin\phi}{\phi'-\phi}, \quad \frac{y}{b}=\frac{2}{3}\frac{\cos\phi-\cos\phi'}{\phi'-\phi}$$

91. CA, CB are the semi-axes of an ellipse, find the position of the centroid of the segment cut off by AB.

92. Generalise by orthogonal projection: P, Q are the points of contact of a common tangent to two circles S_1, S_2; then P, Q are conjugate points w.r.t. any circle coaxal with S_1, S_2.

93. PQ is a diameter of an ellipse, R is any point on the curve; prove that RP, RQ are parallel to a pair of conjugate diameters.

94. CP, CD are conjugate semi-diameters of an ellipse; PVP', DRD' are chords bisected by any other diameter QCQ' at V, R respectively; prove that $CV^2=QR \cdot RQ'$.

95. PQR is a variable triangle inscribed in an ellipse; if the tangents at P, Q are parallel to QR, PR, find the locus of the pole of PQ.

96. PCP', DCD' are two conjugate diameters of an ellipse; CP meets the tangent at the vertex A in H; if $CP=CD$, prove that either PD^2 or PD'^2 equals $2AH^2$.

97. PCP', DCD' are two conjugate diameters of an ellipse; DQ, PR are two parallel chords. Prove that $P'Q$, $D'R$ are parallel to a pair of conjugate diameters.

98. PCP', DCD' are two conjugate diameters of an ellipse; R is any point on the curve; RD, RD' meet PCP' at K, K'; prove that $CK \cdot CK' = CP^2$.

99. Prove that two parallel tangents to an ellipse are met by any other tangent in points situated on conjugate diameters.

100. CP, CD are conjugate semi-diameters of an ellipse; PN, DR are the perpendiculars from P, D to the major axis; prove that (1) $\mathbf{PN^2+DR^2=CB^2}$; (2) $\mathbf{CN^2+CR^2=CA^2}$; (3) $\mathbf{CP^2+CD^2=CA^2+CB^2}$; (4) $\mathbf{PN \cdot NC=DR \cdot RC}$.

101. CP, CD are conjugate semi-diameters of an ellipse; PD cuts the major and minor semi-axes CA, CB at M, N; prove that

$$\frac{CA^2}{CM^2}+\frac{CB^2}{CN^2}=2.$$

102. If P, D are two points on an ellipse, centre C, whose eccentric angles differ by $\frac{\pi}{2}$, prove that CP, CD are conjugate semi-diameters.

103. Prove that there exists one pair of equal conjugate diameters, PCP', DCD' of an ellipse. What are the eccentric angles of their extremities?

[The two equal conjugate diameters of an ellipse are called the *equi-conjugate* diameters.]

104. $PQRS$ is a variable parallelogram inscribed in an ellipse; if its sides are parallel to the equi-conjugate diameters, prove that PQ^2+QR^2 is constant.

105. Through a given point inside an ellipse, show how to draw a straight line so as to divide the ellipse into two parts as unequal as possible.

106. The tangent at any point R of an ellipse, centre C, cuts two conjugate diameters at P, D; prove that the area of the triangle CRP is inversely proportional to the area of CRD.

107. P is a point on an ellipse such that the tangent at P is equally inclined to the semi-axes CA, CB; if the tangent at P cuts the minor axis CB at Q, prove that the area of the triangle $CPQ=\frac{1}{2}CA^2$.

108. From a point T on the semi-diameter CP of an ellipse, TQ is drawn to touch the ellipse at Q; QR is drawn parallel to the semi-diameter CD conjugate to CP and meets CP at R; TD meets QR at V; prove that $QR^2=CD \cdot RV$.

109. The tangents at the extremities P, P' of a diameter of an ellipse meet any other tangent at H, K and any two conjugate diameters at L, M respectively; prove that $PL \cdot P'M=PH \cdot P'K$.

110. If an ellipse is drawn to touch the sides of a triangle at their mid-points, prove that its centre lies at the centroid of the triangle.

111. O is a fixed point; P is a variable point on a given ellipse; Q is a point on OP such that $\frac{OQ}{OP}$ is constant; prove that the locus of Q is a homothetic ellipse.

112. HK is a fixed chord of a given ellipse; P is a variable point on the ellipse; find the locus of the centre of the conic which touches the sides of the triangle HPK at their mid-points.

113. CP, CD are conjugate semi-diameters of an ellipse; the tangent at P meets the major axis CA at T; N is the foot of the perpendicular from P to CA; prove that $\frac{TN}{NC}=\frac{PT^2}{CD^2}$.

114. Generalise by orthogonal projection: If PQ is a diameter of a given circle, and if R is a variable point on the circumference, PR^2+RQ^2 is constant.

115. PQ, PR are two chords of an ellipse; H, K are their respective poles; lines through H, K parallel to PR, PQ meet at F; prove that the centre of the ellipse lies on PF.

116. T is the pole of a chord PQ of an ellipse, centre C; TP, TQ meets CQ, CP at Q', P'; prove that the triangles TPP', TQQ' are equal in area.

117. T is the pole of a chord PQ of an ellipse; a chord HK parallel to TP meets PQ at V, TQ at R; prove that $RV^2=RK\,.\,RH$.

118. [Newton's theorem.] Through a variable point O, two lines of fixed direction are drawn cutting a given ellipse at P, Q and P′, Q′; prove that $\frac{OP\,.\,OQ}{OP'\,.\,OQ'}$ is constant. What special value is obtained by taking O at the centre of the ellipse?

119. Generalise by orthogonal projection: The tangents from any point to a circle are equal.

120. A triangle area Δ, sides a, b, c, is projected orthogonally into an equilateral triangle; prove that the angle between the two planes depends only on the ratio $\frac{\Delta}{a^2+b^2+c^2}$.

121. Two adjacent sides of a parallelogram are of lengths a, b and are inclined at an angle θ; the parallelogram is projected into a square, side x; prove that $2x^2=a^2+b^2-(a^4+2a^2b^2\cos 2\theta+b^4)^{\frac{1}{2}}$.

[Inscribe an ellipse to touch the sides of the parallelogram at its mid-points: what does it project into?]

122. A closed plane curve, area S, is projected on each of three mutually perpendicular planes; σ_1, σ_2, σ_3 are the areas of its projections; prove that $S^2=\sigma_1{}^2+\sigma_2{}^2+\sigma_3{}^2$.

123. Prove that the non-planar curve $x=e^t$, $y=e^{-t}$, $z=t\sqrt{2}$ can be projected orthogonally into a rectangular hyperbola.

124. Prove that the non-planar curve $x=\cos z$, $y=\sin z$ can be projected orthogonally into a circle.

PRACTICAL SOLID GEOMETRY.*

On account of the practical importance of the application of orthogonal projection to problems of solid geometry, it seems desirable, in passing, to call attention very briefly to this aspect of the subject.

Two planes at right angles are taken as planes of reference, which, in their most natural position, would be vertical and horizontal. Projections of points or lines on the horizontal plane (called the H.P.) are named **plans**, and on the vertical plane (called the V.P.) are named **elevations**. In simple cases, the H.P. and V.P. are actually horizontal and vertical; but should this cease to be the case, for convenience of reference, the planes are still called the H.P. and V.P.

The perpendicular from a point to a plane is called the **projector** of that point w.r.t. the plane.

Notation.

The line of intersection of the H.P. and V.P. is called the **ground line**, and is denoted by XY. Capital letters A, B, C, ... denote points in space, their plans are denoted by a, b, c, ..., and their elevations by a', b', c', ... Where two points A, B have the same projection, the point of projection is denoted by $\frac{a}{b}$ or $\frac{a'}{b'}$, if A is nearer to the eye than B, looking at it from above or from the

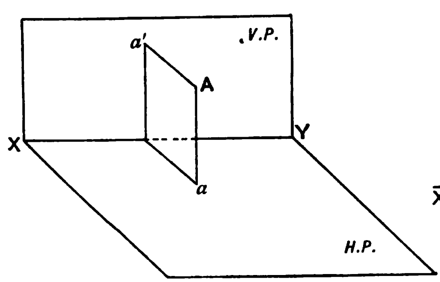

FIG. 17 (1).

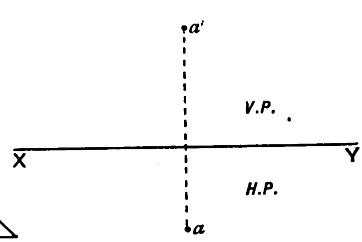

FIG. 17 (2).

front. In the figures of the projections of solids, it is usual to represent visible edges by continuous lines and invisible edges by dotted lines.

* This section (pages 51–64) has been written by Mr. A. E. Broomfield.

THEOREM 35.

(1) The distance of the elevation of a point above XY is equal to the height of the point above the H.P.

(2) The distance of the plan of a point below XY is equal to the distance of the point from the V.P.

(3) The projection of a line, the difference between the lengths of the projectors of its extremities, and the true length of the line, form a right-angled triangle.

[Consequently, given any two of these, it is easy to find the third; see Fig. 7.]

Fig. 17 (2) represents the result of folding, in Fig. 17 (1), the V.P. about XY so as to make its plane coincide with that of the H.P.

125. a, a' are the plan and elevation of A w.r.t. the ground-line XY; prove that aa' is perpendicular to XY.

126. Given the plans and elevations of A, B, construct the length of AB.

127. Given the plan and elevation of a line, construct the angle it makes with the V.P.

128. With the data of Ex. 127, construct the points in which the line meets the H.P. and V.P.

129. What is the condition that the plan of a line is a point?

130. Given the plans and elevations of two lines, what is the condition that the lines are coplanar (*i.e.* intersect)?

131. Given the points in which a line of indefinite length meets the H.P. and V.P., construct the angle it makes with the H.P., and construct its plan.

132. Given the plans and elevations of a point A and a line BC, construct the plan of a line through A, equal and parallel to BC.

133. Given the plan of a triangle ABC, find the plan of its centroid.

134. Given the length of a line AB which meets the H.P. and V.P. at A, B, and given the plan and elevation of the mid-point of AB, construct A and B.

135. abc is the given plan of the triangle ABC; A, B lie in the H.P.; determine the elevation of C on any given V.P., if $A\hat{C}B = 90°$. [If D is the mid-point of AB, note that $AD = CD = BD$.] **Prove that $a\hat{c}b$ must be obtuse.**

SECONDARY PROJECTIONS.

Definition.

The **trace of one plane** on another is their line of intersection; and the **trace of a line** on a plane is their point of intersection.

Thus the ground line, XY, is the trace of the H.P. on the V.P. or the V.P. on the H.P.

THEOREM 36.

Given the plan and elevation of a point, to find the plan on a new H.P., the V.P. remaining unaltered.

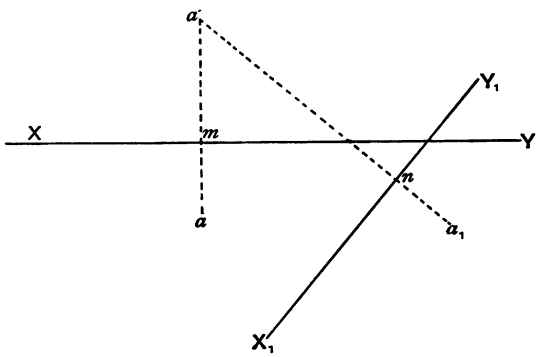

FIG. 18.

Let a, a' be the plan and elevation of the point A w.r.t. the ground line XY; and let X_1Y_1 be the trace of the new H.P. on the original V.P. Let aa' meet XY in m; draw $a'n$ perpendicular to X_1Y_1 and produce it to a_1, so that $na_1 = ma$. Then a_1 and a' are the required plan and elevation w.r.t. the ground line X_1Y_1.

For, since the V.P. is unaltered, the distance of A from the V.P. remains the same, *i.e.* the distance of the plan from its XY is unchanged; consequently, since $a_1n = am$, a_1 must be the required plan. Q.E.F.

It is important to note that, if a line lies in a plane, then the traces of the line lie in the traces of the plane.

136. Given the lines in which a plane meets the H.P. and V.P., and given the plan of a line in this plane, find its elevation.

137. Given the plan and elevation of a point, find the elevation on a new V.P., the H.P. remaining unaltered.

138. Draw the plan and elevation of a cube, which rests with one face on the H.P. and with one edge making a given angle with XY.

139. Draw the plan and elevation of a square pyramid, when the base rests on the H.P. in any given position.

140. Draw the plan and elevation of a square pyramid, which rests with one face on the H.P., if the plan of the axis of the pyramid is (1) parallel, (2) inclined at any given angle, to the V.P.

141. An equilateral triangular prism rests with one side on the H.P.; determine its elevation on any given V.P.

142. A square pyramid rests with its base on the H.P. and one side of the base inclined to XY at 30°; find its plan on a plane, perpendicular to the V.P. and parallel to one edge, if the height of the pyramid is 2″ and a side of the base 1″.

143. Draw the plan and elevation of a regular tetrahedron, of edge 1·5″, which rests with one face on the H.P. in any given position.

144. Draw the plan and elevation of a regular octahedron, which rests with one corner on the H.P., and the diagonal through that corner vertical; the length of its edge being 1″.

145. Draw the elevation of a regular hexagonal prism which rests with one side on the H.P., if its axis makes a given angle with XY.

INCLINATION OF LINES AND PLANES.

The **angle of inclination of two planes** is the angle between the traces in which they are cut by a third plane, perpendicular to their line of intersection.

The **angle of inclination of a line to a plane** is the angle between the line and its projection on that plane.

From this, it follows that whenever *two planes* are represented in projection by two lines, the angle between these lines is equal to the angle between the planes. But if the projections of a plane and a straight line are both straight lines, the angle between them is equal to the angle of inclination of the line to the plane only when the plane of projection is parallel to the line.

THEOREM 37.

To determine the plan and elevation of a cube which has one diagonal inclined to the H.P. at an angle θ.

There are two methods of procedure:

(1) Take the solid, and then choose convenient positions for the planes of reference.

(2) Take two planes of reference, and then arrange the solid in a convenient position.

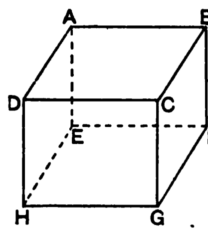

FIG. 19 (1).

In both cases, since we are concerned with the angle of inclination of a line and a plane, we must take one plane of projection parallel to the line.

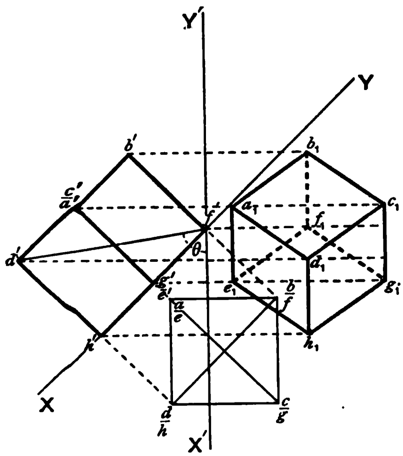

FIG. 19 (2).

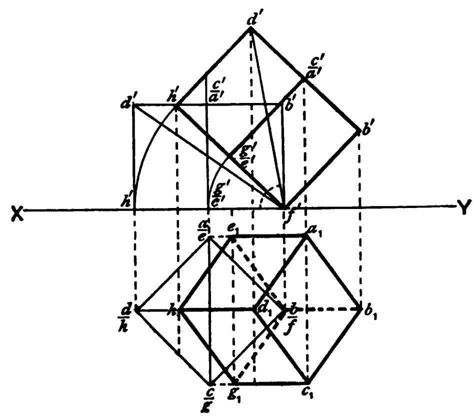

FIG. 19 (3).

Now the simple plan of a cube standing on one face is the square $\frac{a}{e}, \frac{b}{f}, \frac{c}{g}, \frac{d}{h}$; *DBFH* is a vertical plane containing the diagonal *DF*, and is represented by a straight line $\frac{d}{h}, \frac{b}{f}$.

In method (1), if we choose *XY* parallel to $\frac{d}{h}\,\frac{b}{f}$, our V.P. will then be parallel to the plane *DBFH* and therefore parallel to *DF* (Fig. 19 (2)).

In method (2), if we begin with our XY, we must then draw a square, with its diagonal parallel to XY, leading to the same result (Fig. 19 (3)).

In both cases, draw the simple elevation $d'b'f'h'$.

Then either (Fig. 19 (2)), take a new H.P. (ground line $X'Y'$) inclined at the required angle θ to the diagonal or (Fig. 19 (3)), rotate the cube into its required position.

It now remains simply to find the new projections or plans of all the corners in turn.

In Fig. 19 (2), the new plans a_1, b_1, ... must lie respectively on the perpendiculars from a', b', ... to $X'Y'$ and are the same distances from $X'Y'$ as a, b, c, ... are from XY, [Theorem 36]; and are therefore easily constructed. Q.E.F.

In Fig. 19 (3), since the solid is rotated about an axis perpendicular to the V.P., all its corners will rotate in planes parallel to the V.P., and the plans of their paths will therefore be lines parallel to XY.

The points where these parallels cut the perpendiculars from the elevations to XY will therefore be the plans of the corners of the solid, for the position required. Q.E.F.

The second method of obtaining plan and elevation is called **Rabatment**; it will be useful to consider a few more examples of this method; [*e.g.* Theorems 38, 39]. The following facts should be kept prominently in view:

When a plane or solid figure is rotated about a fixed axis, every point of the figure describes a circle whose radius is the distance of the point from the axis.

The planes of all the circles of rotation are parallel to one another and perpendicular to the axis.

Projections of a point, situated on a surface, are often found, conveniently, as the intersection of two lines in the surface, more particularly in problems on sections. This is illustrated by the following example:

EXAMPLE.

Given the plan and elevation of a sphere, and the elevation of a point on its surface, it is required to find the plan of the point.

Let p' be the elevation of the point P.

Take a horizontal section through p'. Its elevation is a horizontal

line through p', cutting the elevation of the sphere in n, say, and a vertical diameter in m.

Then the plan of this section is a circle, of radius equal to mn, and concentric with the plan of the sphere.

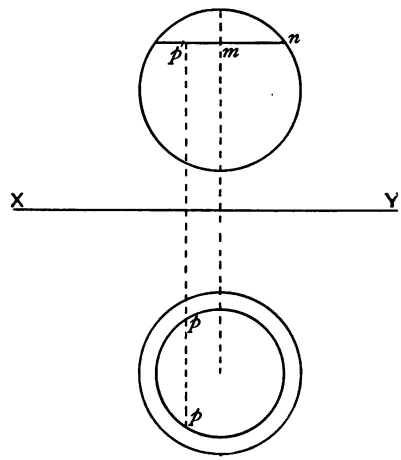

FIG. 19 (4).

The plan p of P lies (1) on the perpendicular from p' to XY, and (2) on the plan of the section, just drawn: and is therefore at one of the points p in the figure. [If the projector from p' be first drawn, the points p may be marked off, without drawing the whole circle in plan.] Q.E.F.

146. A regular hexagonal pyramid rests with one face on the H.P., and the plan of its axis makes an angle of 20° with the V.P.; find its plan and elevation, given that its height is 1″ and the length of a side of the base is ½″.

147. Draw the plan of a circle, radius 2″, if its plane makes an angle of 60° with the H.P. [Construct the plan of a number of points on the circle and join them up free-hand.]

148. A cube rests with one edge in the H.P.; given the angle this edge makes with XY and the angle one face through this edge makes with the H.P., construct its elevation.

149. A regular tetrahedron rests with one edge in the H.P.; given the angle this edge makes with XY and the angle one face through this edge makes with the H.P., construct its elevation.

150. Draw the plan and elevation of a cube, when one diagonal is vertical.

151. Draw the plan and elevation of a square, one edge of which is in the H.P., inclined at a given angle to XY, if the plane of the square makes a given angle with the H.P.

152. Prove that a cube can be made to fit exactly into a regular hexagonal prism. If an edge of the cube is 1″, find a side of the base of the prism.

153. A right prism whose base is an equilateral triangle rests with one face on the H.P. Determine the sectional elevation on any given V.P. [*i.e.* an elevation of the plane surface intercepted by the prism on the given V.P.].

154. A square pyramid rests with one face on the H.P.; determine a sectional elevation on any given V.P. parallel to the plane of section.

155. A cube has one diagonal vertical; draw a sectional plan on a horizontal plane through its centre.

156. The diagonal of a cuboid is vertical; draw a sectional plan on a horizontal plane (1) through its centre, (2) through its lowest corner but two.

157. Given the plan and elevation of a circular cone resting with its base on the H.P., and the plan of a point P on the cone, find the elevation of P.

158. The plan of a square is a given rhombus; find the length of a side of the square.

THEOREM 38.

To draw the plan of a given triangle ABC, situated in the H.P., when its plane is rotated through a given angle θ about a given line PQ, in the H.P.

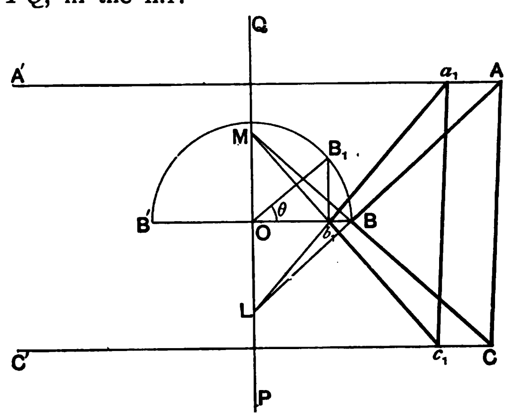

FIG. 20 (1).

The perpendiculars AA', BB', CC', from A, B, C to PQ are the plans of the vertical circles, described by A, B, C. [In the extreme position, $A'B'C'$ is the reflection of ABC in a vertical mirror PQ.]

Let $A_1B_1C_1$ be the new position in space of ABC.

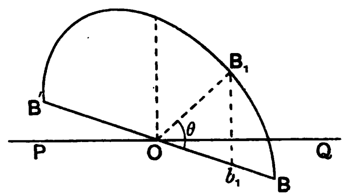

FIG. 20 (2).

The plan of B_1 is the foot b_1 of the perpendicular to BB' from the point B_1, which lies on a vertical semi-circle of diameter BB', centre O, and such that $B_1\hat{O}B = \theta$. This point b_1 is the same point as the foot of the perpendicular from a similar point on *any* semi-circle on BB' as diameter. Draw, therefore, a semi-circle in the plane of the paper, on BB' as diameter; make $B\hat{O}B_1 = \theta$, and draw B_1b_1 perpendicular to BB'.

The plans of A_1 and C_1 and of any other points in the plane may be similarly found.

Much work may often, however, be saved by the use of *stationary points*.

Let AB, CB meet PQ at L, M. Then L, M are stationary points during the rotation.

$\therefore$ Lb_1 produced is the plan of LB_1A_1;

$\therefore$ the plan of A_1 lies on Lb_1 produced.

But the plan of A_1 lies on AA',

$\therefore$ it is the meet a_1 of Lb_1, AA';

and similarly we obtain the plan c_1 of C. Q.E.F.

159. Construct the height of a regular tetrahedron, of edge 1″.

160. Construct the angle which one edge of a regular tetrahedron makes with the plane containing the two edges, concurrent with it.

161. Given three points in the H.P., construct the plan of a point at given distances from these points.

162. *abc* is the given plan of a triangle ABC situated in a given plane perpendicular to the V.P.; if the trace of this plane on the V.P. makes a given angle θ with XY, construct the true size of ABC.

163. Find the trace on the H.P. of a plane inclined at a given angle to the H.P., if its trace on the V.P. is given.

164. Rabat a rectangle through 30° about a diagonal.

165. *abcd* is the given plan of a rectangle; if one diagonal lies in the H.P., find the angle of inclination of the rectangle to the H.P.

166. *P*, *Q* are given points on the sides *AB*, *CD* of a given parallelogram *ABCD*; rabat the parallelogram about *PQ*, through any given angle.

167. Given the traces of a plane on the V.P. and H.P., find its angle of inclination to the H.P.

168. Given the traces of a plane on the V.P. and H.P., draw the plan of a horizontal line, which is at a height of 2″ above the H.P., and lies in the given plane.

169. Given the plan of a triangle ABC which lies in a plane whose traces th, tv on the H.P. and V.P. are given, construct (1) the elevation of ABC, (2) its elevation on a new V.P. perpendicular to th, (3) its horizontal rabatment.

HORIZONTAL PROJECTION.

This is sometimes known as the **Index System**.

Only one plane of projection is used, viz. the horizontal plane.

A **point** is represented by its **indexed plan**,

$$\cdot\, a_{12},$$

which is to be interpreted as the plan of a point, 12 units above the H.P., [the unit generally chosen is $\frac{1}{10}$ inch].

A **line** is represented by the indexed plans of two points on it, as

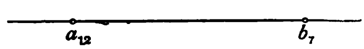

FIG. 21 (1).

A **plane** is represented by the plans of contour lines, abbreviated into what is called a **scale of slope**:

This may be explained as follows:

Let *AB* represent the trace of a plane, perpendicular to the V.P.; and let it be cut by a series of horizontal planes at heights 5, 10, 15, 20, ... units above the H.P. These horizontal lines of section are called **contours**. The plans of these contours are perpendicular to *XY*, and are parallel to one another. They may therefore be completely represented by a scale, drawn at right angles to them, and indexed as in the figure. This is called the scale of slope. It must be remembered that the horizontal trace

of the plane, and the plans of all contours, are perpendicular to the scale of slope.

The scale is formed by two parallel lines, one of which is made thicker than the other. In order to visualise the plane, it

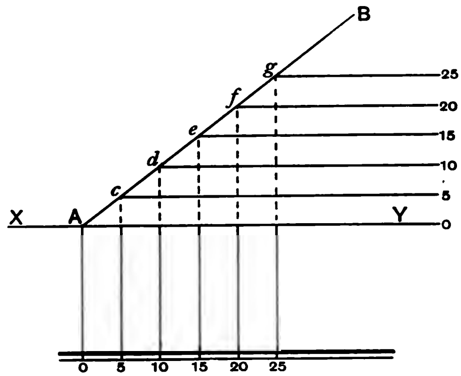

FIG. 21 (2).

is merely necessary to note that *the line, whose indexed plan is represented by the scale of slope, is a line of greatest slope of the plane:* if you stand, looking up the plane, the line of the scale on your left-hand side is the thicker of the two lines forming the scale.

170. Find the indexed plan of a line through a_{10}, equal and parallel to b_7c_{11}.

171. Given the indexed plans of three points A, B, C, find the indexed plan of a horizontal line through A in the plane ABC.

172. Given the indexed plan of a line, find its inclination to the H.P.

173. a_7, b_{12}, c_{14}, are the indexed plans of three vertices of a parallelogram; determine the index of the remaining vertex.

174. Given the indexed plans of two lines, find the condition that the lines are coplanar.

175. Given the scale of slope of a plane, construct its angle of inclination to the H.P.

176. Given the scale of slope of a plane, and the plan of a point in it, find its index.

177. Construct the scale of slope of a plane, parallel to a plane of given scale of slope and passing through a point of given indexed plan.

178. Find the point of intersection of a plane of given scale of slope with a line of given indexed plan.

179. Given the scale of slope of a plane, find the plan of a line in the plane which passes through a given point and makes a given angle with the H.P.

THEOREM 39.

To find the true shape of the triangle ABC, when its indexed plan, $a_8b_{11}c_{16}$, is given.

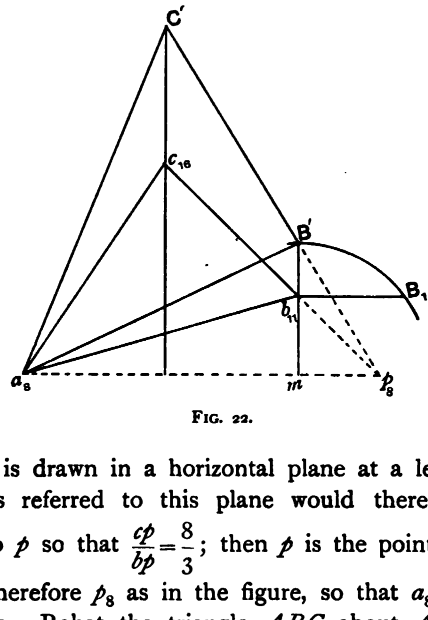

FIG. 22.

The figure is drawn in a horizontal plane at a level 8, and the indexed plans referred to this plane would therefore be $a_0b_3c_8$. Produce cb to p so that $\frac{cp}{bp}=\frac{8}{3}$; then p is the point of level 8 on BC, and is therefore p_8 as in the figure, so that a_8p_8 or AP is a horizontal line. Rabat the triangle ABC about AP into a horizontal position $AB'C'$. To do this, draw bm perpendicular to ap, and bB_1 perpendicular to bm and equal in length to the difference of level of B and m, *i.e.* 3.

Now mb is the plan of mB, and bB_1 is equal to the difference of level of B and m; therefore, by Theorem 35 (3), mB_1 equals mB and is the true length of mb_{11}.

But, when the figure is rotated about a_8p_8, B traces out a circle, centre m, in a plane perpendicular to a_8p_8. Consequently the circle, centre m, radius mB_1, cuts mb at the required point B'. [Th. 38.]

Now p_8 is a stationary point on BC,

$\therefore$ $B'C'$ must pass through p_8.

Therefore $B'p_8$ meets the perpendicular from c to a_8p_8 at the required point C'.

Then $aB'C'$ is the true shape of the triangle $a_8b_{11}c_{16}$. Q.E.F.

THEOREM 40.

To find the plan of the line of intersection of two planes, given by their scales of slope.

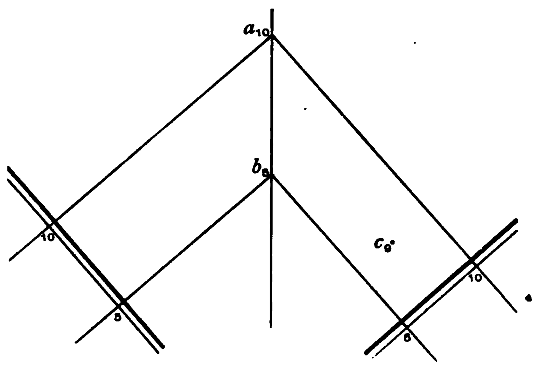

FIG. 23.

Draw through the plans of the '5' contours (or two other contours of equal level) in each plane, perpendiculars to the respective scales of slope.

These contours (1) lie in the same horizontal plane, and therefore either intersect or are parallel; and (2) lie one on each of the given planes,

Therefore, from (1) they have a point of intersection, and from (2), this point of intersection lies in both of the given planes.

Therefore the plans of these contours meet at the plan of a point on the line of intersection of the given planes.

Another point can similarly be found, thus giving the indexed plan of the required line of intersection. Q.E.F.

The case where the scales of slope of the two planes are parallel is omitted and left as an exercise to the reader, see Ex. 182.

180. In Fig. 23, is c_9 above or below the plane whose scale of slope is given on the right hand side of the figure?

181. With the notation of Ex. 180, obtain the length of the perpendicular from c_9 to the plane.

182. Give a construction for Theorem 40, if the scales of slope of the planes are parallel.

183. Construct the scale of slope of a plane, which passes through three given points a_5, b_7, c_{13}. [Find a point d_7 on a_5b_{13}.]

184. Given the indexed plans of two points A, B, and the plan of a point C, find its index if AC, BC are equally inclined to the H.P.

185. Given the scales of slope of three planes, find the indexed plan of their common point.

186. A northward path on the plane face of an embankment rises 3 feet vertically in a horizontal distance of 4 feet, and a westward path rises 5 feet in 12 feet; find the direction of the steepest possible path, and its inclination to the horizontal.

187. Draw a plane making a given angle with the H.P. and passing through a line of given indexed plan.

188. Given the scales of slope of three planes, find the indexed plan of the centre of a sphere, of given radius, touching the three planes. Is there more than one solution, and if so, how many?

189. The floor of a room, which has a plane sloping ceiling, is a given horizontal irregular quadrilateral; its walls are vertical; given the heights of three of the top corners, find the height of the remaining top corner.

190. A pyramid stands with its base on the H.P.; given its plan and elevation, as also that of a line through one corner of the base, find where this line cuts the pyramid again.

191. Given the plan and elevation of a sphere, and a line parallel to both planes of reference, find the points of intersection of the line with the sphere. [Take an auxiliary plan w.r.t. a ground line, perpendicular to the given ground line.]

192. Given the indexed plan of two lines AB, AC, find the plan of the line bisecting the angle BAC.

CHAPTER III.

CONICAL PROJECTION.

THE idea of conical projection is first found, in an elementary form, in the writings of Serenus (circa 450 A.D.). But no real use was made of it before the time of Desargues (1593-1662), a French architect and engineer, who served under Richelieu at the siege of Rochelle. The originality of his ideas, and the fertility of his methods, place him among the greatest geometers of all time. The modern theory of projective geometry is only a development of the principles which characterise his researches: and it is interesting to notice that the non-metrical *Geometry of Position* of Von Staudt takes as its starting-point Desargues' property of perspective triangles. His work was not however appreciated at its true value by his contemporaries, with the exception of Pascal; chiefly because the new analytical field of discovery, opened out by Descartes, appeared more attractive and more far-reaching in its consequences. And it was left to Carnot (1753-1823), Poncelet (1788-1867), and Chasles (1793-1880) to perceive its merits and develop its principles. By his wonderful discovery of "the circular points at infinity," and their connection with the foci, and by his enunciation of the fundamental "Principle of Duality," Poncelet was enabled to coordinate and generalise the theory of conics, elevating it from a collection of independent properties to a connected and logical unity. The conceptions he introduced into geometry were extended by Plücker, a professor at Bonn, who was the first to *define* the foci of a conic by their isotropic characteristic, and so build up by analogous methods a theory of foci for curves of higher degree: while by his invention of tangential coordinates (1829), he supplied the analytical basis for Poncelet's method of reciprocal polars.

Definitions.

(1) $P_1, P_2, \ldots$ are a system of points in a plane Σ. O is any fixed point outside Σ; the lines OP_1, $OP_2, \ldots$ meet a second given plane σ at the points $p_1, p_2, \ldots$. Then the system of points $p_1, p_2, \ldots$ is said to be the **conical projection** of the given system in Σ on σ w.r.t. the point O, which is called the **vertex of projection.**

(2) The line of intersection of Σ and σ is called the **axis of projection.**

(3) If a plane through O parallel to σ meets Σ in the line L, then L is called the **vanishing line** of Σ: and similarly, if a plane through O parallel to Σ meet σ in the line m, then m is called the **vanishing line** of σ.

An example of conical projection is supplied by the magic lantern. Each detail of the slide is projected by the source of light on to the screen: and the picture thus formed is the conical projection of the slide. To form an accurate representation, the planes of the screen and slide must be parallel: otherwise the picture is a distortion of the original, *i.e.* metrical properties, the sizes of angles and the ratios of lengths of lines, are altered. Such a distortion consequently is usually produced by projection. But invariably each point and each line in the picture correspond uniquely to a point and a line in the slide, and the joins of corresponding points are concurrent: and it is these two features which are the essential characteristics of conical projection.

We proceed to enumerate a number of simple properties connecting figures and their projections.

Unless otherwise stated, capital letters will refer to the given figure, and small letters to the corresponding elements of the projected figure.

THEOREM 41.

(1) A straight line projects into a straight line.

(2) The meet of two straight lines projects into the meet of their projections.

(3) The join of two points projects into the join of their projections.

(4) Any point on the axis of projection is unaltered in position by projection.

(5) Any straight line and its projection meet on the axis of projection.

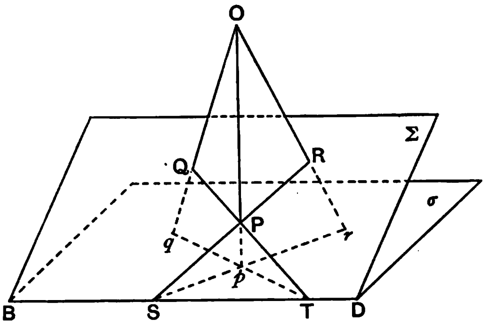

FIG. 24.

The proof is left to the reader.

THEOREM 42.

(1) The cross ratio of four collinear points is equal to the cross ratio of their projections: and in particular a harmonic range projects into a harmonic range.

(2) The cross ratio of four concurrent lines is equal to the cross ratio of their projections: and in particular a harmonic pencil projects into a harmonic pencil.

The proof is left to the reader.

THEOREM 43.

(1) If a system of concurrent lines meet at a point on the vanishing line of their plane, then their projections form a system of parallel lines.

(2) Each point on the vanishing line projects into an ideal point or point at infinity, in the plane of the projected figure.

(3) The vanishing line projects into an ideal line, called the line at infinity, in the plane of the projected figure.

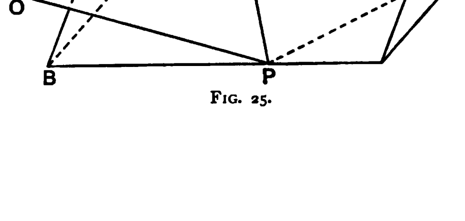

FIG. 25.

(1) Let Σ, σ be the two planes, and O the vertex of projection.

The vanishing line L of Σ is the intersection of Σ with a plane λ through O, parallel to σ.

Let D be the point on L at which the system of lines concur.

Let any line DP of the system meet the axis of projection at P.

Then by definition, the projection of DP is the line of intersection of the planes ODP, σ.

Now λ and σ are parallel planes.

$\therefore$ the plane ODP cuts λ, σ in parallel lines.

$\therefore$ the projection of DP is parallel to OD.

$\therefore$ the system of concurrent lines project into a system of lines parallel to OD. Q.E.D.

(2) Now, by definition, a system of parallel lines determine an ideal point or point at infinity, common to each member of the system.

Therefore the projection of the point D determined by a system of concurrent lines is that ideal point determined by the system of parallel lines into which they project. Q.E.D.

(3) Further, the line at infinity is defined as the aggregate of all ideal points of the plane, determined by taking all possible systems of parallel lines. But every point on the vanishing line projects into an ideal point. Therefore the vanishing line projects into the line at infinity. Q.E.D.

The proof of this theorem could be rendered in another way:

To any point on the vanishing line, there does not correspond any (finite) point in the second plane. Consequently a system of lines meeting at a point on the vanishing line project into a system of lines having no (finite) point of intersection, and so, by definition, form a system of parallel lines: further this point of concurrence on the vanishing line may be regarded as projecting into the ideal point determined by the system of parallels.

It would, however, be illogical, in view of the definition of infinity previously given, to start by saying that the vanishing line projects into the line at infinity, and that therefore lines concurring at a point of the vanishing line project into parallel lines. This is an inversion of the proper order of thought. For the vanishing line is said to project into the line at infinity, only because lines concurring at a point of it project into parallel lines.

Definition.

The meet of any line of Σ with the vanishing line of Σ is called the **vanishing point** of that line.

THEOREM 44.

(1) If D is the vanishing point of the line PD, and if O is the vertex of projection, the projection of PD is parallel to OD.

(2) If H, K are the vanishing points of two lines QH, QK; and if O is the vertex of projection, the angle between the projections of QH, QK, is equal to the angle HOK.

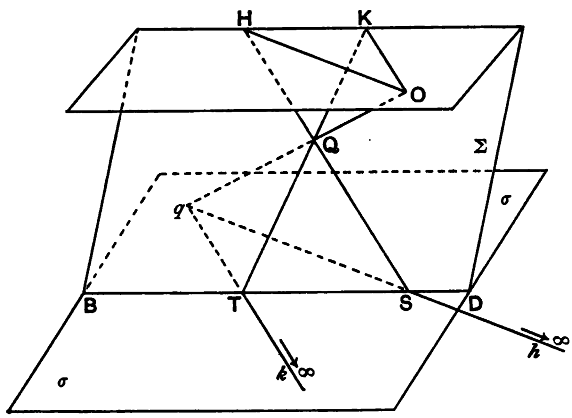

FIG. 26.

(1) This has been already proved in the course of Theorem 43.

(2) Denoting elements of the projected figure by small letters, we have qh, qk are respectively parallel to OH, OK.

$\therefore$ the angle qh makes with qk is equal to the angle HOK.

Q.E.D.

THEOREM 45.

Given any geometrical system in a plane Σ and a vertex of projection O, it is possible to find a plane σ such that any given line PQ in Σ projects into the line at infinity in σ.

Take for σ any plane parallel to the plane OPQ; then PQ is the vanishing line of Σ, and therefore projects into the line at infinity in σ. Q.E.D.

It is convenient to express more briefly the process of this theorem by saying that PQ is to be projected to infinity.

The use of conical projection may be illustrated at this stage by the following example.

EXAMPLE.

To establish the harmonic property of the quadrilateral.

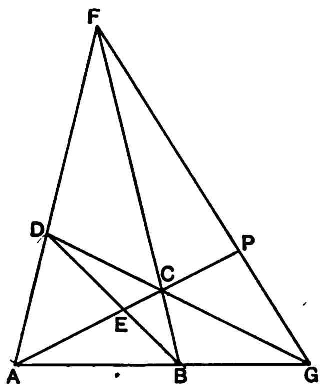

FIG. 27.

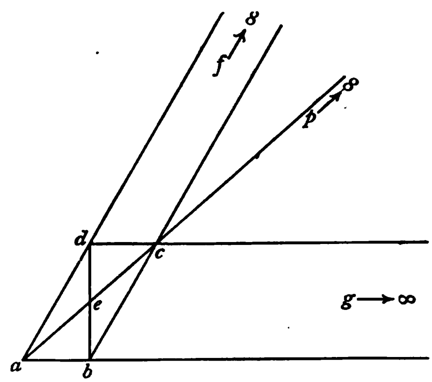

FIG. 28.

With the notation of Fig. 27, it is required to prove that $\{AECP\}$ is harmonic.

Project FG to infinity: Fig. 28 represents the projected system.

AD, CB meet at F; $\therefore$ ad, cb are parallel.

DC, AB meet at G; $\therefore$ dc, ab are parallel.

$\therefore$ $abcd$ is a parallelogram;

$\therefore$ $ae = ec$.

Since P lies on FG, p is a point at infinity;

$\therefore$ $\{aecp\}$ is harmonic;

$\therefore$ $\{AECP\}$ is harmonic. Q.E.D.

THEOREM 46. [DESARGUES' THEOREM.]

ABC, $A'B'C'$ are two triangles in the same or different planes. If AA', BB', CC' are concurrent, then the meets L, M, N of BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ are collinear.

(1) Let ABC, $A'B'C'$ lie in two different planes Σ, Σ'; and let AA', BB', CC' concur at O.

Then $B'C'$, $C'A'$, $A'B'$ are the projections of BC, CA, AB w.r.t. O on Σ'.

$\therefore$ by Theorem 41 (5), BC meets $B'C'$ on the axis of projection.

$\therefore$ L, and similarly M, N, lie on the axis of projection, *i.e.* the line of intersection of Σ, Σ'. Q.E.D.

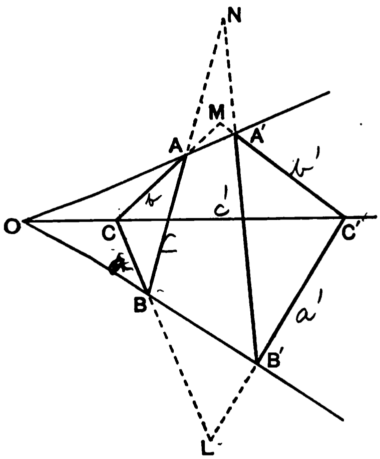

FIG. 29.

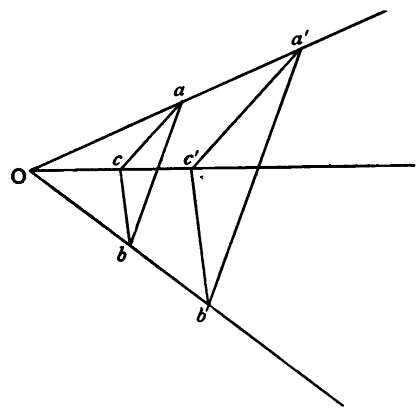

FIG. 30.

(2) Let ABC, $A'B'C'$ lie in the same plane.

Project LM to infinity.

BC, $B'C'$ meet at L; $\therefore$ bc, $b'c'$ are parallel.

CA, $C'A'$ meet at M; $\therefore$ ca, $c'a'$ are parallel.

$$\therefore \frac{ob}{ob'} = \frac{oc}{oc'} = \frac{oa}{oa'};$$

$\therefore$ ab, $a'b'$ are parallel;

$\therefore$ AB, $A'B'$ meet on LM;

$\therefore$ L, M, N are collinear. Q.E.D.

THEOREM 47. [DESARGUES' THEOREM.]

ABC, $A'B'C'$ are two triangles in the same or different planes. If the meets L, M, N of BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ are collinear, then AA', BB', CC' are concurrent.

(1) Let ABC, $A'B'C'$ lie in different planes.

Since BC, $B'C'$ intersect, BB', CC' lie in a plane, α say.

Similarly, let β, γ, be the planes of $CC'AA'$, $AA'BB'$.

Then AA', BB', CC' are the lines of intersection of the pairs of planes β, γ; γ, α; α, β.

But any three planes α, β, γ have one common point, so that their lines of intersection with each other are concurrent;

$\therefore$ AA', BB', CC' are concurrent. Q.E.D.

(2) Let ABC, $A'B'C'$ lie in the same plane.

Project LMN to infinity.

The proof is left to the reader.

1. Prove Theorem 41.

2. Prove Theorem 42.

3. Prove Theorem 47 (2).

4. Prove that any three collinear points can be projected into two points and the mid-point of their join.

If the order of points on the line is A, B, C, draw a figure to show how C may be projected into the mid-point of the projection of AB.

5. Prove that three collinear points A, B, C can be projected into three points a, b, c so that $\frac{ab}{bc}$ has a given value.

6. Given a point P and a triangle ABC, prove that it is possible to project P into the centroid of the projection of ABC.

7. Prove that the vanishing line of either plane Σ or σ is parallel to the axis of projection.

8. Prove that three concurrent lines can be projected into three parallel equidistant lines.

9. A straight line meets the sides BC, CA, AB of a triangle at P, Q, R; P', Q', R' are the harmonic conjugates of P, Q, R w.r.t. BC, CA, AB respectively; prove that AP', BQ', CR' are concurrent.

10. Generalise by projection: the mid-points of the three diagonals of any quadrilateral are collinear.

11. P, Q, R are points on the sides BC, CA, AB of a triangle, prove that the value of $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}$ is unaltered by projection.

12. Generalise the last exercise, so as to obtain a theorem for any polygon.

13. Use Ex. 11 to prove Ceva's theorem, by projecting one of the three concurrent lines to infinity.

14. Use Ex. 11 to prove Menelaus' theorem by projecting the transversal to infinity.

15. ABCDEF are six collinear points : l, m, n are any constants. If l . AB . CD . EF + m . AC . BE . DF + n . AD . BF . CE = 0, prove that this relation is unaltered by projection.

[This is a generalisation of the theorem that the cross ratio of any four collinear points is unaltered by projection.]

16. AB is the axis of projection ; CD is the vanishing line of Σ ; $A'B'$ is the reflection of AB in CD ; prove that the length of any segment of $A'B'$ is unaltered by projection.

17. Prove from first principles, by projection, that the third diagonal of a complete quadrilateral is divided harmonically by the two other diagonals.

18. A, B, C ; A', B', C', are two sets of three collinear points ; prove that the meets of AB', $A'B$; BC', $B'C$; CA', $C'A$ are collinear.

19. O is the vertex, and AB the axis, of projection ; CD, ef are the vanishing lines of Σ, σ ; prove that the distance of O from CD is equal to the distance of AB from ef.

20. A, B, C are three fixed collinear points; PQR is a variable triangle such that P, Q lie on fixed lines and QR, RP, PQ pass through A, B, C respectively; prove that the locus of R is a straight line.

21. Each of the vertices of a variable quadrangle lies on a fixed line, the four fixed lines being concurrent : and three of the sides pass through fixed points, every vertex being situated on at least one of these sides ; prove that the remaining three sides pass through fixed points.

22. Generalise Ex. 21, to obtain a result for any polygon.

23. Generalise by projection :

$ABCD$ is a parallelogram ; through any point O lines are drawn parallel to the sides to meet AB, CD, BC, AD at P, Q, H, K ; then PK, BD, HQ are concurrent.

24. Generalise by projection :

With the notation of Ex. 23, AQ, KC, OB are concurrent.

25. Any line meets the sides AD, DC, CB, BA of a quadrilateral at P, Q, R, S ; P', Q', R', S' are the harmonic conjugates of P, Q, R, S w.r.t. AD, DC, CB, BA ; prove that PQ, $P'Q'$, $R'S'$ are concurrent.

26. Three triangles are such that their vertices lie on the same three concurrent straight lines, prove that the axes of perspective of the triangles, taken in pairs, are concurrent.

27. Three triangles are such that, when taken in pairs, they are in perspective and have the same axis of perspective ; prove that the three centres of perspective of the triangles, taken in pairs, are collinear.

28. AA', BB', CC' are three concurrent lines ; prove that the six meets of AB, AB', AC, AC', BC, BC' with $A'B'$, $A'B$, $A'C'$, $A'C$, $B'C'$, $B'C$ respectively are the vertices of a complete quadrilateral.

THEOREM 48.

It is possible to project two pairs of lines into pairs of lines containing angles of given magnitudes, and at the same time any other given line into the line at infinity.

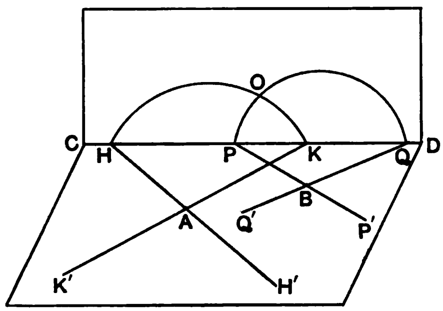

FIG. 31.

Let HAH', KAK'; PBP', QBQ' be the two pairs of lines which are to be projected into pairs of lines containing angles α, β respectively. Let the given line CD meet the other lines at H, K, P, Q.

In any other plane, through CD, describe on HK, PQ segments of circles containing angles α, β: and let O be one of the points of intersection of these circles.

With O as vertex, project the system on to any plane parallel to the plane OCD.

Then, by Theorem 44, since H, K, P, Q are vanishing points, it follows from the construction that the projected pairs of lines contain angles α, β: and the line CD is projected to infinity.

Q.E.D.

It should be noted that the two circles may not intersect at real points. In such a case the vertex of projection is imaginary. Since however the process of projection corresponds to a definite analytical transformation (see p. 76), it may still be regarded as valid, in accordance with the Principle of Continuity. Any results obtained by making use of such a projection will still be true even if the process ceases to have any graphical significance. The employment of geometrical language is merely a convenient means of describing a particular analytical operation.

THEOREM 49.

Given a straight line PQ and a triangle ABC, it is possible to project ABC into a triangle similar to a given triangle, and at the same time the line PQ to infinity.

The proof is left to the reader.

THEOREM 50.

If a line APB touches a curve S at P, then the projection of APB touches the projection of S at the projection of P.

The tangent at P is the limiting position of the chord PQ, when $Q \to P$.

And the tangent at p is the limiting position of pq, when $q \to p$.

$\therefore$ the tangent at p is the projection of the tangent at P.

Q.E.D.

29. Prove Theorem 49.

30. P is a variable point in Σ; H, K are two fixed points on the vanishing line of Σ; prove that $h\hat{p}k$ is constant.

31. Prove that a quadrilateral can be projected into a rhombus, having one diagonal equal to a side.

32. Prove that a quadrilateral can be projected into a square of given size.

33. Prove that any three concurrent lines and a transversal can be projected into three parallel lines and a transversal perpendicular to them and cut into two equal segments.

34. If two curves touch, prove that their projections touch.

35. What is the projection of a common tangent to two curves?

36. Prove that two quadrilaterals which have a common third diagonal can be projected into rhombuses.

37. XY is the common line of two planes Σ, σ; any point O is taken as vertex of projection; α, β are the planes through XY, which bisect the angle between Σ, σ; the perpendiculars from O to α, β meet Σ at M, N and σ at m, n. If A, B are any two points in Σ, and a, b their projections in σ, prove that $A\hat{M}B = a\hat{m}b$.

[Produce MA, MB to meet XY at A', B'; rotate Σ about XY until it coincides with σ and show that MA', MB' then coincide with mA', mB'.]

38. (1) ABCD are four collinear points; P, Q are a pair of points harmonically conjugate to B, C and to A, D; prove that the locus of points at which AB, CD subtend equal angles is the circle on PQ as diameter.

(2) Hence show how to project any three angles into three equal angles and any line to infinity.

39. Show how to project any five points into five concyclic points, and any line to infinity. [Use Ex. 38.]

40. Prove that two triangles in perspective can be projected into two triangles, each similar to a given triangle.

41. Prove that two curves S_1, S_2 intersecting at A, B can be projected into two curves s_1, s_2 cutting each other orthogonally at a, b, and any line to infinity.

42. Find the locus of the vertex of projection w.r.t. which a given quadrilateral can be projected into a square.

ANALYTICAL TREATMENT OF CONICAL PROJECTION.

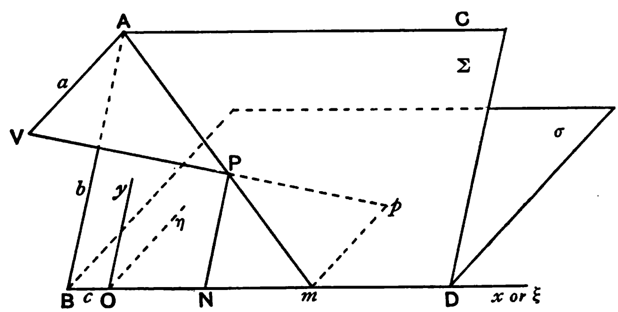

FIG. 32.

V is the vertex, and BD the axis, of projection.

The plane through V parallel to σ meets Σ in the vanishing line AC.

The plane through V perpendicular to BD meets BD, AC at B, A.

P is any point in Σ, VP meets σ at p.

Take as origin any point O on BD.

For axes take, in Σ, Ox, Oy along and perpendicular to OD; and in σ, $O\xi$, $O\eta$ along and perpendicular to OD.

Let (x, y), (ξ, η) be the coordinates of P, p referred to these axes.

Let AP meet BD at m.

Now the plane VPA meets the parallel planes VAC, σ in parallel lines; therefore VA, pm are parallel.

But VA is perpendicular to BD, since the plane VAB is perpendicular to BD; therefore pm is perpendicular to BD.

Draw PN perpendicular to BD.

Let $VA=a$, $AB=b$, $BO=c$.

By parallels,
$$\frac{pm}{VA}=\frac{Pm}{PA}=\frac{Nm}{NB};$$
$$\therefore\ \frac{\eta}{a}=\frac{\xi-x}{x+c},$$
which gives
$$x=\frac{a\xi-c\eta}{\eta+a}.$$
Similarly,
$$\frac{PN}{AB}=\frac{mP}{mA}=\frac{mp}{mp+VA};$$
$$\therefore\ \frac{y}{b}=\frac{\eta}{\eta+a}\quad\text{or}\quad y=\frac{b\eta}{\eta+a}.$$

Therefore, by a proper choice of axes, any figure and its conical projection are related by the equations
$$x=\frac{a\xi-c\eta}{\eta+a};\quad y=\frac{b\eta}{\eta+a}.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

If the origin is taken at B, since $c=0$, these assume the simpler form
$$x=\frac{a\xi}{\eta+a};\quad y=\frac{b\eta}{\eta+a}.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II.)}$$

These equations represent an analytical transformation, which corresponds to the geometrical operation of projection. The constants a, b, c may be regarded as defining the position of the vertex of projection. If a, b, c are all real, the vertex of projection is real, and the operation is purely geometrical: but if any of these constants are imaginary, V is an imaginary point. This does not however invalidate the analytical process. For projection might be defined as the effect of the transformation (I.); then if a, b, c are real, it admits of a graphical interpretation, which was taken, on p. 66, as the geometrical definition of projection. If a, b, c are imaginary, the analysis is in no way affected; and if it leads to results which are capable of geometrical expression, the theorems so formulated must be true. By using terms in a wider significance, it is possible, as has been already explained, to indicate, in geometrical form, an analytical process which has no graphical analogue.

And in this way we are enabled to avoid a laborious piece of analysis. Any two conics can, for example, be projected into two circles by a *real* projection, if they do not cut at real points. Consequently any descriptive property connecting the two circles can be transformed into a property of the two conics by purely geometrical methods: but this transference could be effected with equal logic by analysis, although usually with considerably less ease. If, however, the two conics have four real points of intersection, it is impossible to obtain a *real* projection which will change them into circles. Consequently the graphical transference is no longer possible. But analysis makes no distinction between the two cases, and therefore the transmitted property must still hold good.

Again, in Theorem 48, it was proved that two angles could be projected into angles of given size, and also a given line to infinity, by taking the vertex of projection at a point of intersection of two known circles. If these circles do not meet at real points, the vertex of projection is no longer real: but it is convenient still to speak of projecting w.r.t. the imaginary point of intersection as vertex, although the phrase now refers to nothing more than an analytical transformation, the constants a, b, c of which are obtained from a consideration of the equations of the two circles in question. In other words, the geometrical language affords a simple means of describing a rather long and tedious piece of analysis, the result of which can be foreseen from purely geometrical arguments. The analysis is indeed an essential part of the proof: but since it leads inevitably to a result that can be predicted, it is usual to omit it, with the assurance that it could be effected; and refer to the omission under the name of conical projection. The notation used in the following sections, unless otherwise stated, is that of Fig. 32.

THEOREM 51.

(1) Every curve of the second degree is a conic.

(2) One conic, and one only, can be drawn through five points, no four of which are collinear.

(1) Any curve of the second degree can be represented by an equation of the form, $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Project this, according to the transformation

$$x=\frac{p\xi}{\eta+p},\quad y=\frac{q\eta}{\eta+p},\quad \text{(see II. on p. 77)}.$$

The curve becomes

$$ap^2\xi^2 + 2hpq\xi\eta + bq^2\eta^2 + (2gp\xi + 2fq\eta)(\eta + p) + c(\eta + p)^2 = 0.$$

This is a circle, if the coefficients of ξ^2 and η^2 are equal, and if the coefficient of $\xi\eta$ is zero.

This will be the case, if $ap^2 = bq^2 + 2fq + c$ and $2hpq + 2gp = 0$; which require that $q = -\frac{g}{h}$ and $p^2 = \frac{1}{ah^2}(bg^2 - 2fgh + ch^2)$.

Choosing the transformation in this way, we see that the curve of the second degree projects into a circle and is therefore by definition (see p. 88) a conic. Q.E.D.

(2) The general equation of the second degree contains five *independent* constants, which enter linearly; they can therefore be chosen so as to make the curve pass through five given points, and this, in only one way, if no four of the points are collinear.

Q.E.D.

(A) Lines which intersect on the vanishing line project into parallel lines.

The equation of the vanishing line AC is $y = b$.

Any system of lines intersecting on $y = b$ can be represented by $px + qy + \lambda(y - b) = 0$, where λ varies.

Putting $$x = \frac{a\xi}{\eta + a}; \quad y = \frac{b\eta}{\eta + a},$$

this becomes $$pa\xi + qb\eta - \lambda ab = 0$$

which represents a system of parallel lines, as λ varies. Q.E.D.

(B) The degree of a curve is unaltered by projection.

This is immediately evident by substitution.

(C) A system of concentric circles project into a system of conics having double contact with each other: and the line at infinity projects into the chord of contact.

Any circle of the system can be represented by

$$(x - f)^2 + (y - g)^2 = r^2,$$

where r varies.

Therefore the equation of its projection is

$$[a\xi - f(\eta + a)]^2 + [b\eta - g(\eta + a)]^2 = r^2(\eta + a)^2,$$

which represents a system of conics touching each other at their meets with $\eta + a = 0$, which, either by analysis or on referring to Fig. 32, is at once seen to be the projection of the line at infinity.

Q.E.D.

(D) A system of conics having double contact with each other at two fixed points E, F can be projected into a system of concentric circles; and the line EF will be projected to infinity.

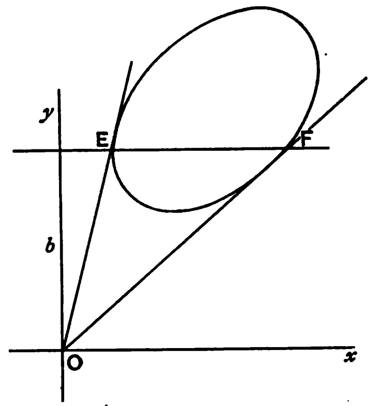

FIG. 33.

Take the meet O of the common tangents at E, F, [*i.e.* the pole of EF] as origin, and the x-axis, parallel to EF.

Let $y=b$ be the equation of EF.

Any conic of the system is represented by

$$px^2+2qxy+ry^2-\lambda(y-b)^2=0,$$

where λ varies.

Apply to this conic the transformation $x=\dfrac{a\xi-c\eta}{\eta+a}$, $y=\dfrac{b\eta}{\eta+a}$.

[This is chosen because it makes $y-b=0$ a vanishing line.]

The conic becomes

$$p(a\xi-c\eta)^2+2q(a\xi-c\eta)b\eta+rb^2\eta^2-\lambda(b\eta-b\eta-ba)^2=0,$$

or $$pa^2\xi^2-2a\xi\eta(pc-qb)+\eta^2(pc^2-2qbc+rb^2)-\lambda a^2b^2=0.$$

This is a circle if $pc=qb$ and $pa^2=pc^2-2qbc+rb^2$, which give

$$c=\frac{qb}{p},\quad a^2=b^2\frac{pr-q^2}{p^2}.$$

And the conic becomes $\xi^2+\eta^2=\dfrac{\lambda b^2}{p}$, which is a circle, with centre at the origin, *i.e.* the pole of EF. Q.E.D.

It is worth while enquiring under what conditions the projection is real.

We suppose that the equations of the conics have real coefficients.

Since $c=\frac{qb}{p}$, c is always real.

Since $a^2=b^2\frac{pr-q^2}{p^2}$, a is real, only if $pr-q^2$ is positive; which is the condition that the lines $px^2+2qxy+ry^2=0$ are imaginary. But these two lines are OE, OF; and are therefore imaginary if O lies inside the conics.

Hence the projection is real or imaginary, according as the points of contact of the conics are imaginary or real.

If $pr-q^2=0$, the lines OE, OF are coincident, and the general equation of the conic breaks up into linear factors.

(E) A system of conics through four fixed points can be projected into a system of circles: and the line joining two of the points will be projected to infinity. Further, the projection is real if at least two of the common points are imaginary, the conics being real.

The proof is left to the reader.

[By a suitable choice of axes, any conic of the system can be represented by $px^2+2qxy+ry^2+\lambda(y-b)(lx+my+1)=0$, where λ varies.]

Analytical methods are to be used in the following exercises.

43. Prove Theorem (E).

44. Prove that any conic can be projected into a circle and at the same time any given line to infinity.

45. Prove that any conic can be projected into a circle having the projection of a given point as centre.

46. Prove that the cross ratio of four concurrent lines is unaltered by projection.

47. Prove that a point and its polar w.r.t. a conic project into a point and its polar w.r.t. the projection of the conic.

48. Prove that the centre of a conic S in the plane Σ projects into the pole of the vanishing line in σ w.r.t. the conic s.

49. Prove that a conic touching the vanishing line projects into a parabola.

50. Prove that any two angles can be projected into angles of given size, and at the same time any given line can be projected to infinity.

51. If two variable lines make with two fixed lines a pencil of constant cross ratio, prove that the variable lines can be projected into lines including a constant angle.

52. Prove that a given point can be projected into a given point and a given line to infinity.

53. Prove that three concurrent lines can be projected into three parallel equidistant lines.

54. Can a given conic be projected into a circle, any given point being the vertex of projection?

55. Prove that a system of conics having the same focus and directrix can be projected into concentric circles.

56. The conics $2x^2+3y^2=1$; $x^2=2y$ are projected into circles; find the necessary equations of transformation.

57. The conics $3x^2+2y^2=1$; $y^2=2x$ are projected into circles; find the necessary equations of transformation.

58. The conics $x^2-2y+1=0$; $2x^2-y^2=2y-1$ are projected into circles; find the necessary equations of transformation: and prove that the circles are concentric.

59. Find a transformation by which the conics

$$y^2=3x-4; \quad x^2-y^2-3x+4=0$$

may be projected into circles.

60. If in Fig. 32 the origin o is at B, and if ox makes an angle θ with BD, prove that the equations of transformation are

$$x=\frac{a\cos\theta.\xi+b\sin\theta.\eta}{\eta+a}; \quad y=\frac{-a\sin\theta.\xi+b\cos\theta.\eta}{\eta+a};$$

where $o\xi$ is, as before, along BD.

AREAL COORDINATES.

The use of areal coordinates supplies another simple analytical treatment of projection.

Definition.

If ABC is a fixed triangle, called the triangle of reference, and if P is a variable point in the plane, the ratios

$$\frac{\triangle BPC}{\triangle BAC}, \quad \frac{\triangle CPA}{\triangle CBA}, \quad \frac{\triangle APB}{\triangle ACB}$$

are called the **areal** coordinates of P w.r.t. the triangle; and will be denoted by ξ, η, ζ; where the area of any triangle XYZ is reckoned positive if the direction $X\to Y$, $Y\to Z$, $Z\to X$ is anti-clockwise. Thus, for example, $\triangle XYZ+\triangle XZY=0$.

It follows at once, from the definition, that the areal coordinates ξ, η, ζ of any point are invariably connected by the relation

$$\xi+\eta+\zeta=1.$$

THEOREM.

Ox, *Oy*, *Oz* are three mutually perpendicular lines; any plane cuts them at *A*, *B*, *C*. *P* is any point in the plane; x, y, z are the coordinates of *P* referred to *Ox*, *Oy*, *Oz*, [*i.e.* x is the length of the perpendicular from *P* to the plane *yOz*, etc.]; ξ, η, ζ are the areal coordinates of *P* w.r.t. the triangle *ABC*. If $OA=a$, $OB=b$, $OC=c$, then $x=a\xi$, $y=b\eta$, $z=c\zeta$; and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

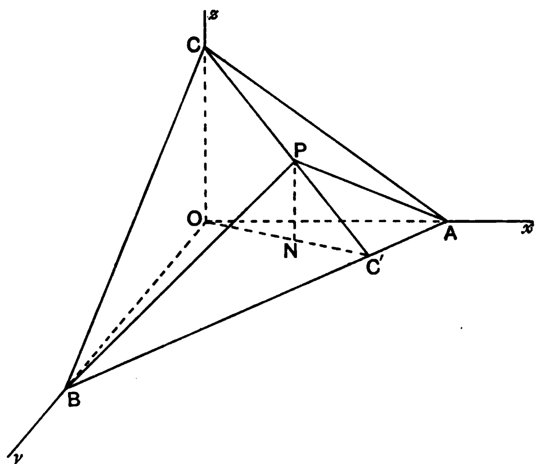

FIG. 34.

Let *CP* cut *AB* at *C′*. Draw *PN* perpendicular to *OC′*; since the plane *OCC′* is perpendicular to the plane *BOA*, *PN* is perpendicular to the plane *BOA*; $\therefore$ $PN=z$.

$$\therefore \frac{z}{c}=\frac{PN}{CO}=\frac{PC'}{CC'}=\frac{\triangle BPA}{\triangle BCA}=\zeta;$$

$$\therefore z=c\zeta; \text{ and similarly } x=a\xi,\ y=b\eta.$$

Q.E.D.

But $\xi+\eta+\zeta=1$; $\therefore$ $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Q.E.D.

It should be noted that, as a result of this theorem,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

may be called the equation of the plane *ABC*.

This theorem enables us to obtain the equations of transformation for conical projection.

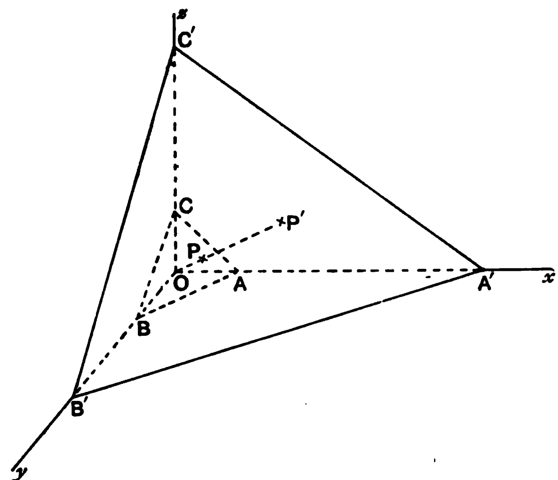

FIG. 35.

Through the vertex O of projection, draw three rectangular axes Ox, Oy, Oz cutting the plane Σ at A, B, C, and the plane Σ', on to which the figure is to be projected, at A', B', C'.

Let $OA = a$, $OA' = a'$, etc.

Let ξ, η, ζ be the areal coordinates of P w.r.t. the triangle ABC; and let ξ', η', ζ' be the coordinates of the projection P' of P w.r.t. $A'B'C'$.

Let x, y, z; x', y', z' be the coordinates of P, P' w.r.t. ox, oy, oz.

Then $\qquad \dfrac{x}{x'} = \dfrac{y}{y'} = \dfrac{z}{z'}$; but $x = a\xi$, $x' = a'\xi'$, etc.

$$\therefore \frac{a\xi}{a'\xi'} = \frac{b\eta}{b'\eta'} = \frac{c\zeta}{c'\zeta'}; \text{ also } \xi + \eta + \zeta = 1;$$

$$\therefore \frac{\xi}{\frac{a'}{a}\xi'} = \frac{\eta}{\frac{b'}{b}\eta'} = \frac{\zeta}{\frac{c'}{c}\zeta'} = \frac{1}{\frac{a'}{a}\xi' + \frac{b'}{b}\eta' + \frac{c'}{c}\zeta'}$$

which is the required transformation. Q.E.D.

It is evident that the analysis is affected very slightly if any fixed point is taken as origin, instead of the vertex of projection.

In conclusion, we propose to describe the geometrical connection between a curve given by an equation in Cartesians, and the curve

whose equation is of the same form, in areals. For example, what connection, if any, exists between the curve $x^2+y^2=1$ and the curve $\xi^2+\eta^2=\zeta^2$?

The reader, who is acquainted with the theory of homogeneous coordinates, is aware that a large number of formulae, proved in the first instance for Cartesians, apply equally to generalised coordinates. The equation of a tangent, the condition for a harmonic pencil, etc., are examples of forms which remain unaltered. In fact, descriptive properties in general are unaffected by the change of coordinates. It is therefore natural to expect that the connection is of a projective nature.

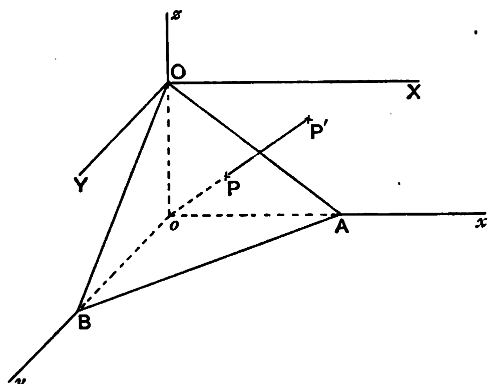

FIG. 36.

Let the plane $x+y+z=1$ cut ox, oy, oz at A, B, O; draw OX, OY parallel to ox, oy.

Let x_1, y_1, z_1 be the coordinates of any point P in the plane $x+y+z=1$ w.r.t. ox, oy, oz: and let ξ, η, ζ be the areal coordinates of P w.r.t. the triangle ABO.

Let oP cut the plane XOY at P': and let X_1, Y_1 be the coordinates of P' w.r.t. OX, OY.

Then $$\frac{x_1}{X_1}=\frac{y_1}{Y_1}=\frac{z_1}{1}.$$

Also $$x_1=\xi_1,\ y_1=\eta_1,\ z_1=\zeta_1, \text{ by the theorem on p. 83;}$$

$$\therefore\ \frac{\xi_1}{X_1}=\frac{\eta_1}{Y_1}=\frac{\zeta_1}{1}.$$

Therefore, if ξ_1, η_1, ζ_1 traces out the curve $f(\xi,\ \eta,\ \zeta)=0$, then X_1, Y_1 traces out the curve $f(X,\ Y,\ 1)=0$.

Therefore the curve $f(\xi, \eta, \zeta)=0$ may be regarded as the conical projection of the curve $f(x, y, 1)=0$, which establishes the connection mentioned above.

61. Determine a geometrical connection between the curve $x^2+y^2=r^2$, (Cartesians), and the curve $a^2\xi^2+b^2\eta^2=c^2r^2\zeta^2$, (areals).

62. What is the geometrical connection between the two curves $\xi^2+\eta^2+\zeta^2=0$; $a\xi^2+b\eta^2+c\zeta^2=0$? (areals).

63. With the notation of Fig. 36, prove that the projection of

$$\frac{u}{\xi}+\frac{v}{\eta}+\frac{w}{\zeta}=0$$

(which represents any conic circumscribing ABO) w.r.t. o, on the plane $z=1$, is a rectangular hyperbola. Explain this also on geometrical grounds.

64. With the notation of Fig. 36, prove that the projection of

$$\sqrt{u\xi}+\sqrt{v\eta}+\sqrt{w\zeta}=0$$

(which represents any conic inscribed in ABO) w.r.t. O, on the plane $z=1$, is a parabola. Explain this also geometrically.

65. In Fig. 36, the position of any point P in the plane OBA is given; obtain a geometrical construction to determine the point in which oP meets the plane XOY, which is parallel to xoy.

66. In Fig. 35, the position of any point P in the plane ABC is given, obtain a geometrical construction to determine the point in which OP meets the plane $A'B'C'$.

67. AB represents the line of intersection of two planes; AC, AD represent lines of greatest slope in the two planes; V is a point outside both planes; P is a given point in the plane CAB; obtain a geometrical construction for the point in which VP meets the plane DAB.

CHAPTER IV.

THE CONIC.

To find the first mention of the conic, it is necessary to go back as far as the fourth century B.C. Its discovery is attributed to a disciple of Plato, named Menaechmus (350–330 B.C.), who employed it to solve the famous Delian problem, known as the Duplication of the Cube. His researches were, however, very fragmentary, being probably restricted to the barest elements of the parabola $y^2 = ax$, and the rectangular hyperbola $xy = c^2$, with its asymptotes. The earliest writer known to have regarded the conic as a section of a cone was Aristaeus (circa 320 B.C.); while the first systematic treatment was given by Euclid (323–284 B.C.), in a book now lost. This formed the basis of the famous Κωνικά of Apollonius (247–205 B.C.), which deservedly gained him among the ancients the title of the "Great Geometer." It contains a wonderfully complete account of the (non-focal) properties of the conic, its conjugate diameters and asymptotes, and includes the harmonic property of the pole and polar, for the case in which the pole lies outside the curve—a theorem which was completed only after the lapse of eighteen centuries by Desargues. The sense of Continuity, which had been introduced into Geometry by Kepler, illustrated for example by his view that the parabola has a centre, which is a point at infinity, was developed by the genius of Desargues. By using the idea of the line at infinity, and the cognate notion of parallelism, he showed that the asymptotes could be regarded as tangents at infinity, the centre as the pole of the line at infinity, and conjugate diameters as a special case of conjugate lines w.r.t. the conic. To him is also due the harmonic theory of the quadrangle inscribed in a conic.

The principal value of the process of projection is the link it supplies between the nature of a circle and a conic, affording, as it does, a rapid means of generalising a complete category of properties of the circle, by transmitting them to the conic. Much of the power of this method is due to the cross-ratio theory of the conic, which is identified with the name of the famous French Geometer, Chasles (1793–1880). It is most remarkable that the fundamental theorem [Theorem 57], which is a simple deduction from Pappus' theorem (p. 163), and Apollonius' theorem on the cross-ratio property of four concurrent lines, should have remained unnoticed for another two thousand years. The method adopted in the present chapter is due to Chasles, who, acting on the suggestion of M. Delbalat, one of his former pupils, was the first to point out the ease with which the descriptive properties of the conic can be developed from this basis. For more detailed historical information, the reader is referred to Chasles' *Aperçu Historique*, and Dr. Taylor's *Ancient and Modern Geometry of Conics*.

Definition.

Any curve formed by the projection of a circle is called a **conic**.

There are three kinds of conics.

(1) The conic is called a **hyperbola**, if the line at infinity cuts it at **real** points.

If the vertex of projection is real, this case arises when the vanishing line cuts the generating circle at real points.

(2) The conic is called a **parabola**, if the line at infinity touches the conic.

This will happen if the vanishing line touches the generating circle.

(3) The conic is called an **ellipse**, if the line at infinity cuts it at **imaginary** points.

If the vertex of projection is real, this case arises when the vanishing line cuts the generating circle at **imaginary** points.

This definition may be put in another form: Imagine a cone, whose base is a circle, and vertex O; let a plane L cut the cone. Then the curve obtained on L is a conic. If the plane through O, parallel to L, meets the base-circle in imaginary points [see Fig. 37], the conic is a closed curve and is called an ellipse; if the plane through O, parallel to L, touches the base circle, the

conic is an open curve, with one branch, and is called a parabola; if the plane through O, parallel to L, meets the base circle at real

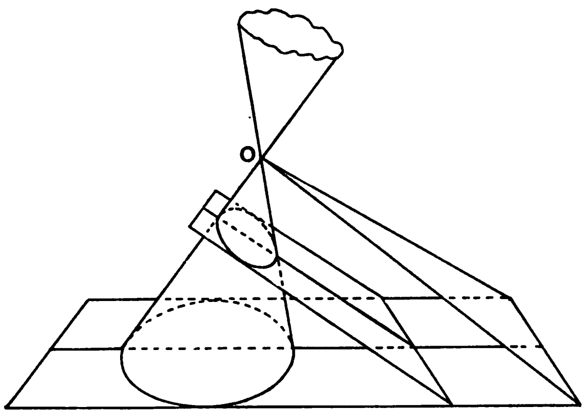

FIG. 37.

points, the conic is an open curve with two branches, and is called a hyperbola.

Any circle is cut by any straight line in two and only two points which may be real, coincident, or imaginary [see p. 3]. And from any point, there can be drawn to any circle two, and only two tangents, which may be real, coincident or imaginary. Both these properties are easily proved by analysis: and owing to the analytical definition of imaginary elements adopted in Chapter I., any method of proof must fundamentally be of an algebraic character.

It follows, therefore, that a conic, as defined above, is a curve of the second degree (*i.e.* every line meets it at two and only two points), and of the second class (*i.e.* from every point two and only two tangents can be drawn to it). Moreover it has been proved, in Theorem 51, that every curve of the second degree is necessarily a conic: and it may be shown, fairly easily, by analysis that every curve of the second class must be a curve of the second degree (see p. 202), and therefore a conic. In treatises dealing with the metrical properties of conics, the focus-directrix property is usually adopted as the definition. This has undeniable advantages, but it does not agree with the historical development of the subject, as is pointed out at the beginning of Chapter V., which deals with the theory of the foci, from which this important property may be very simply deduced [p. 136].

Definition.

If a variable line is drawn through a fixed point P, meeting a fixed conic Σ at H, K, and if Q is the harmonic conjugate of P w.r.t. H, K; then the locus of Q is called the **polar** of P w.r.t. Σ and P is called the **pole** of the locus of Q.

THEOREM 52.

(1) If the conic Σ is the projection of the circle σ, any pole and polar w.r.t. Σ is the projection of a pole and polar w.r.t. σ.

(2) The polar of a point P w.r.t. a conic Σ is a straight line which passes through the points of contact, real or imaginary, of the tangents from P to Σ.

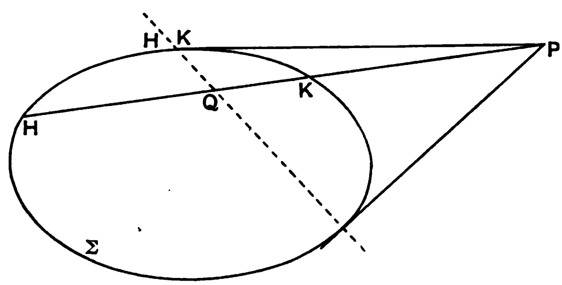

FIG. 38.

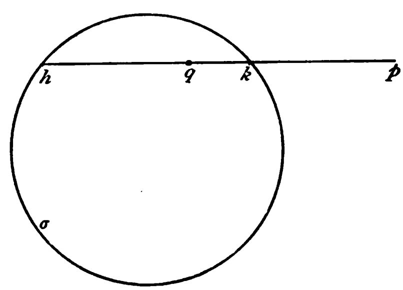

FIG. 39.

(1) Let a variable line through P cut Σ at H, K: and let Q be the harmonic conjugate of P w.r.t. H, K. Then, by definition, the locus of Q is the polar of P.

With the usual notation for the elements in the plane of σ, since $\{HK; PQ\}$ is harmonic, $\{hk; pq\}$ is also harmonic.

Therefore since p is a fixed point, q traces out the polar of p.

Therefore the polar of P w.r.t. Σ is the projection of the polar of p w.r.t. σ. Q.E.D.

(2) The polar of P, being the projection of a straight line, viz. the polar of p w.r.t. σ, is also a straight line.

Now, when PHK touches Σ, H coincides with K at the point of contact: and therefore Q coincides with both.

Therefore the points of contact of the tangents from P to Σ lie on the polar locus.

Now this proof, from a geometrical standpoint, applies only to the case where P lies outside Σ: but since the theorem admits of an analytical treatment, which takes no account of whether P lies inside or outside Σ, it follows that the theorem is true for all positions of P. [This is simply a statement of the Principle of Continuity.] Q.E.D.

THEOREM 53.

(1) If the polar of a point P w.r.t. a conic passes through a point Q, then the polar of Q passes through P.

(2) If the pole of a line p lies on a line q, then the pole of q lies on p.

The proof is left to the reader.

[Regard the conic as the projection of a circle. See Part I., page 159, Theorem 72.]

Definitions.

(1) Two points such that the polar of either, w.r.t. a conic, passes through the other, are called **conjugate points** w.r.t. the conic.

(2) Two lines such that the pole of either, w.r.t. a conic, lies on the other, are called **conjugate lines** w.r.t. the conic.

(3) If A, B, C are the poles of the sides of the triangle PQR w.r.t. a conic, the triangles ABC, PQR are called **conjugate triangles** w.r.t. the conic.

(4) If the vertices of a triangle are the poles of the opposite sides w.r.t. a conic, the triangle is called a **self-conjugate triangle** w.r.t. the conic.

THEOREM 54.

Two conjugate lines, which meet at T, are harmonically conjugate to the tangents from T to the conic; and conversely, two lines, which are harmonically conjugate to the tangents from a point T to a conic, are conjugate lines w.r.t. it.

The proof is left to the reader.

[See Part I., page 165, Theorem 74.]

THEOREM 55.

The cross ratio of the ranges formed by any four collinear points is equal to the cross ratio of the pencil formed by their polars w.r.t. any conic.

The proof is left to the reader.

[In Part I., page 165, Theorem 75, the two pencils are equiangular.]

THEOREM 56.

T is the pole of a chord AB of a conic: the tangent at any other point C meets TA, TB, AB at H, K, D; then $\{HK;\ CD\}$ is harmonic.

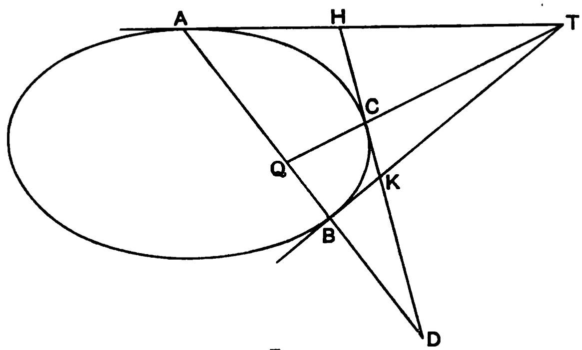

FIG. 40.

Let TC cut AB at Q.

The polar of T (*i.e.* AB) passes through D.

$\therefore$ the polar of D passes through T.

But the polar of D passes through C, and is therefore TC.

$\therefore$ by the definition, $\{AB;\ QD\}$ is harmonic.

$\therefore$ $T\{AB;\ QD\}$ is a harmonic pencil.

$\therefore$ $\{HK;\ CD\}$ is a harmonic range. Q.E.D.

NOTE ON METHOD.

The following considerations will often suggest a way in which to look for a solution of a given problem. The application of some of the following remarks will not be understood, however, until a further portion of the chapter has been read.

If P, Q, R are collinear points, and if it is required to prove that $PQ = QR$, it may be simplest to show that $\{PQR\infty\}$ is

harmonic, where ∞ denotes the ideal point on PQ; and more generally, to prove that $\frac{PQ}{QR}$ is constant, it may be easiest to prove that $\{PQR\infty\}$ is of constant cross ratio; or again, $PQ.PS=PR^2$, if $\{QPR\infty\}=\{RPS\infty\}$.

To prove two or more lines parallel, it may be useful to employ the idea that they concur at a point at infinity, or that their poles w.r.t. some conic lie on a diameter. To prove a range is harmonic, it is sometimes possible to connect it by a pencil with a harmonic system of points on a conic, which are formed by any two conjugate chords. [Theorem 58.]

If a property is concerned with the asymptotes of a hyperbola, it may be suggestive to draw an ideal figure containing the asymptotes as actual tangents, with the line at infinity drawn in as their chord of contact.

Again, in the case of the parabola, an ideal figure, in which the line at infinity is represented as an actual tangent to the curve at a point, through which every diameter must pass, is sometimes helpful. Or, instead of regarding such a figure as ideal, it may be taken to represent a projection of the actual figure, with the line at infinity projected into a finite line. If the property is of a projective nature, this new figure then indicates a more general theorem, of which the given property is a particular case; and this general theorem is frequently easier to establish, because the irrelevant details of the figure have been removed.

1. Prove Theorem 53.

2. Prove Theorem 54.

3. Prove Theorem 55.

4. P, Q are two conjugate points w.r.t. a conic; R is the pole of PQ; prove that PQR is a self-conjugate triangle.

5. A variable chord PQ of a conic passes through a fixed point; prove that the tangents at P, Q meet on a fixed line.

6. TA, TB are the tangents from a variable point T on a fixed line to a given conic; prove that AB passes through a fixed point.

7. D is a given point on the base BC of a fixed triangle ABC; prove that the polar of D w.r.t. any conic touching AB, AC at B, C is a fixed line.

8. T is the pole of a chord PQ of a conic; if the portion of another tangent intercepted by TP, TQ is bisected at its point of contact, prove that it is parallel to PQ.

9. QR is a fixed chord of a conic; P is a variable point on the conic; prove that the harmonic conjugate of the tangent at P w.r.t. PQ, PR passes through a fixed point.

10. The hypotenuse BC of a right-angled triangle cuts a conic at H, K; if B is the pole of AC, prove that AC bisects $H\hat{A}K$.

11. PQR is a self-conjugate triangle w.r.t. a conic; if P lies inside the conic, prove that the chord through P parallel to QR is bisected at P.

12. Given a point P and a conic, prove that an unlimited number of self-conjugate triangles PQR can be drawn.

13. PA, PB are two conjugate lines w.r.t. a conic; the polar of a point R on PA meets PA, PB at A, B and the conic at H, K; prove that $\{AB; HK\}$ is harmonic.

14. L_1, L_2, L_3, L_4 are four lines forming a harmonic pencil; S is a conic touching L_1, L_3, at A, B; prove that the polar of any point on L_2 w.r.t. S is concurrent with AB, L_4.

15. OA, OB are two chords of a conic equally inclined to the tangent at O; prove that the pole of AB lies on the normal at O.

16. T is a variable point and A a fixed point on a fixed line AT; P, Q are the points of contact of the tangents from T to a given conic S; prove that the polar of A w.r.t. any conic touching S at P, Q passes through a fixed point.

17. T is the pole of a chord PQ of a conic; the bisector of the angle PTQ meets PQ at H; RS is any chord through H, prove that TH bisects the angle RTS.

18. $ABCD$ is a quadrangle; AB, CD meet at E; AC, BD meet at G; prove that a conic drawn through G to touch AD, BC at A, B will touch EG.

19. Three conics are drawn through a common point D to touch AB, AC at B, C; BC, BA at C, A; CA, CB at A, B; prove that the tangents at D to the conics meet BC, CA, AB respectively in three collinear points.

20. T is the pole of a chord PQ of a conic; the tangent at any other point K of the conic cuts TP at H; QK meets TP at L; prove that $\{TH; PL\}$ is harmonic.

21. A variable chord PQ meets a fixed chord RS of a given conic at a fixed point O; the tangents at P, Q cut RS at H, K; prove that $\frac{1}{OH}+\frac{1}{OK}$ is constant, taking account of the sense of the lines.

22. Two tangents to a conic at A, B meet at right angles at D; the tangent at any other point P meets AB, AD at Q, R; prove that DR bisects $P\hat{D}Q$.

23. T is the pole of a chord PQ of a conic; the bisector of $P\hat{T}Q$ meets PQ at H; prove that the pole of any other chord through H lies on the other bisector of $P\hat{T}Q$.

24. A conic touches the sides BC, CA, AB of a triangle at P, Q, R; prove that AP, BQ, CR are concurrent.

25. ABC is a triangle inscribed in a conic; prove that the tangents at its vertices meet the opposite sides in three collinear points.

26. A conic cuts the sides BC, CA, AB of a triangle at A_1, A_2; B_1, B_2; C_1, C_2; if AA_1, BB_1, CC_1 are concurrent, prove that AA_2, BB_2, CC_2 are concurrent.

27. A variable plane through a fixed line L cuts a fixed cone in the conic σ; prove that the locus of the pole of L w.r.t. σ is a straight line.

28. B, C are conjugate points w.r.t. a conic S; P is any point on S; BP, CP cut S again at Q, R; prove that QR passes through the pole of BC.

29. ABC, PQR are two triangles self-conjugate w.r.t. the same conic; PQ, PR cut BC at Q', R'; AB, AC cut QR at B', C'; prove that

$$RB'C'Q\} = R'BCQ'\}.$$

30. T is the pole of a chord PQ of a circle Σ, which is projected into a conic σ, PQ being the vanishing line. Prove that the projection t of T is a point such that every chord through t of σ is bisected at t.

31. With the notation of Ex. 30, what will TP, TQ project into?

32. AC, BD are conjugate chords of a conic; AD meets the tangent at B in L; AB meets the tangent at D in M; N is the pole of AC; prove that L, M, N are collinear.

33. Two chords PQ, RS of a conic are conjugate w.r.t. the conic; any chord RT meets PQ at H; if $P\hat{S}Q = 90°$, prove that SP, SQ are the bisectors of $H\hat{S}T$.

THEOREM 57. [CHASLES' THEOREM.]

A, B, C, D are four fixed points on a conic; a, b, c, d are the tangents to the conic at these points. P is a variable point on the conic: t is a variable tangent to the conic. Then the

cross ratio of the pencil $P\{ABCD\}$ is constant: and the cross ratio of the range $t\{abcd\}$ is constant: and the values of these two cross ratios are equal.

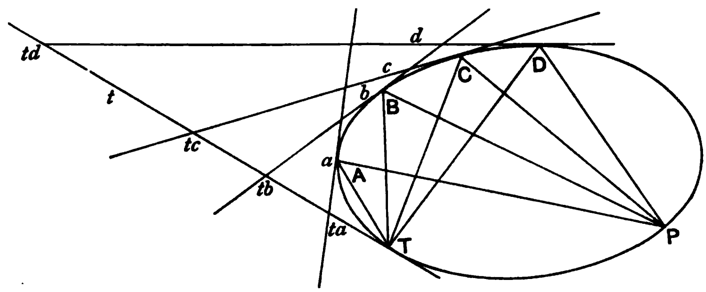

FIG. 41.

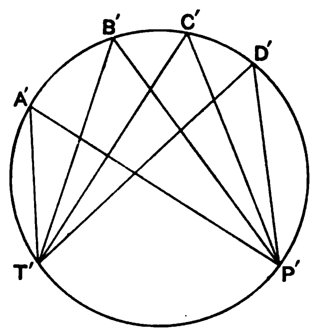

FIG. 42.

Regard the conic as the projection of a circle.

Denote by dashes corresponding points in the plane of the circle.

If T is the point of contact of t, the pencils $P'\{A'B'C'D'\}$, $T'\{A'B'C'D'\}$ are equiangular [*i.e.* the corresponding angles are either equal or supplementary] and therefore equicross.

∴ the pencils $P\{ABCD\}$, $T\{ABCD\}$ are equicross.

∴ the pencil $P\{ABCD\}$ is of constant cross ratio. Q.E.D.

Again the pole of TA is the meet of t, a, *i.e.* the point ta.

∴ by Theorem 55, the range $t\{abcd\}$ is equicross with the pencil $T\{ABCD\}$ or $P\{ABCD\}$, and is therefore constant.

Q.E.D.

And the values of these two cross ratios $t\{abcd\}$, $P\{ABCD\}$ are therefore equal. Q.E.D.

Definition.

(1) If A, B, C, D are four points on a conic, such that $P\{ABCD\}$ is a harmonic pencil, where P is any other point on the conic, then A, B, C, D are called a **harmonic system of points** on the conic.

(2) If a, b, c, d are four tangents to a conic, such that $p\{abcd\}$ is a harmonic range, where p is any other tangent to the conic, then a, b, c, d are called a **harmonic system of tangents** to the conic.

THEOREM 58.

(1) If two chords AC, BD are conjugate lines w.r.t. the conic; then A, B, C, D form a harmonic system of points on the conic.

(2) Conversely, if A, B, C, D form a harmonic system of points on the conic, then AC, BD are conjugate lines w.r.t. the conic.

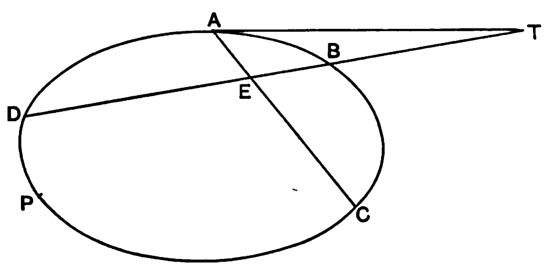

FIG. 43.

(1) Let the tangent at A meet BD at T: and let P be any point on the conic.

Then the pole of AC lies both on BD and AT and therefore is T.

$$\therefore \{TBED\} \text{ is harmonic.}$$

Now $$P\{ABCD\} = A\{ABCD\} \qquad \text{[Th. 57]}$$
$$= A\{TBED\};$$
$$\therefore P\{ABCD\} \text{ is harmonic.} \qquad \text{Q.E.D.}$$

(2) The proof of the converse is left to the reader.

34. Prove Theorem 58 (2).

35. P, Q are two conjugate points w.r.t. a conic, prove that the tangents from P, Q to the conic form a harmonic system of tangents to the conic.

THEOREM 59.

T is the pole of a chord MN of a conic; any line cuts the conic at P_1, P_2 and TM, TN, MN at Q_1, Q_2, R; then

$$Q_1P_1 . P_2R . RQ_2 = -Q_2P_2 . P_1R . RQ_1.$$

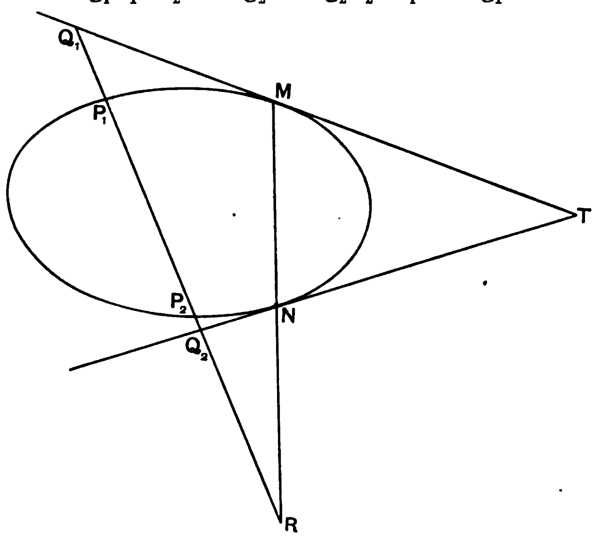

FIG. 44.

By Chasles' Theorem, $M\{MP_1P_2N\} = N\{MP_1P_2N\}$.

$$\therefore\ M\{Q_1P_1P_2R\} = N\{RP_1P_2Q_2\};$$

$$\therefore\ \frac{Q_1P_1 . P_2R}{Q_1R . P_2P_1} = \frac{RP_1 . P_2Q_2}{RQ_2 . P_2P_1};$$

$$\therefore\ Q_1P_1 . P_2R . RQ_2 = -Q_2P_2 . P_1R . RQ_1. \quad \text{Q.E.D.}$$

This is a special case of a more general theorem, due to Desargues (see p. 298).

36. Prove that Theorem 56 is a special case of Theorem 59.

37. ABC is a triangle inscribed in a conic; a chord PQ meets BC, CA, AB at L, M, N; and the tangent at P meets BC at T; prove that $\{TLBC\} = \{PQNM\}$.

38. PQ, PQ' are two chords of a conic equally inclined to the normal chord PP' at P; prove that $P'Q$, $P'Q'$ are harmonically conjugate w.r.t. $P'P$ and the tangent at P'.

39. O is the pole of a chord BC of a conic; OQP is a line cutting the conic at P, Q; BA is a chord parallel to OP, prove that AC bisects PQ.

40. PP', QQ' are conjugate chords of a conic; a line through P parallel to the tangent at Q cuts QQ', QP' at H, K; prove that $PH=HK$.

41. Two conics S_1, S_2 touch each other at B, C; A is the pole of BC; P, Q, R, S are four points on AB, the tangents from which to S_1, S_2 meet AC at P_1, Q_1, R_1, S_1 and P_2, Q_2, R_2, S_2; prove that $\{P_1Q_1R_1S_1\}=\{P_2Q_2R_2S_2$

42. ABCDE are five points on a given circle, prove that

$$\mathbf{A\{BCDE\}=\frac{BC\,.\,DE}{BE\,.\,DC}}.$$

43. PP', QQ' are conjugate chords of a conic; a line through P' parallel to QQ' cuts the conic at R; PR cuts QQ' at H, prove that $QH=HQ'$.

44. ABC is a triangle inscribed in a conic; $\mathbf{C_1}$ is the pole of AB; any line through $\mathbf{C_1}$ cuts BC, AC at M, N; prove that M, N are conjugate points w.r.t. the conic.

45. PQ is a chord of a conic; O is any point on PQ; M is any point on the polar MN of O; a parallel through O to MQ cuts MP, MN at B, A; prove that $OB=BA$.

46. LMN is a triangle inscribed in a conic; if LN and the normal at M are conjugate lines, prove that LM, NM are equally inclined to the normal at M.

47. OP, OP' are two chords of a conic, equally inclined to the chord OB; if OA is the chord perpendicular to OB, prove that PP' passes through the pole of AB.

48. T is the pole of a chord MN of a conic; a chord PQ of the conic parallel to TN meets TM, MN at H, R, prove that

$$HR^2=HP\,.\,HQ.$$

49. A conic is inscribed in a parallelogram; prove that any other tangent to the conic is cut harmonically by the sides of the parallelogram.

50. PQ is a chord of a conic bisecting another chord AB at O; the tangents at P, Q meet AB at S, T; prove that $AS=BT$.

51. A conic touches the sides of a quadrilateral. Deduce from Chasles' theorem a property by taking the variable tangent as coinciding successively with the sides of the quadrilateral.

52. A variable tangent to a conic cuts two fixed parallel tangents, whose points of contact are P, P' at M, M'; prove that $PM\,.\,P'M$ is constant.

[Use the idea of Ex. 51.]

53. A is the pole of a chord BC of a conic; a chord QR of the conic cuts AB, AC, BC at M, N, P; prove that $\dfrac{MR\,.\,NR}{MQ\,.\,NQ}=\dfrac{PR^2}{PQ^2}$.

54. A, B, C, D, P are five points on a conic; α, β, γ, δ represent the lengths of the perpendiculars from P to AB, BC, CD, DA; prove that $P\{ABCD\}=\frac{\alpha.\gamma}{\beta.\delta}\cdot\frac{AB.CD}{AD.CB}$.

55. [Pappus' Theorem.] A, B, C, D are four fixed points on a conic; P is a variable point on the conic; α, β, γ, δ are the lengths of the perpendiculars from P to AB, BC, CD, DA, prove that $\frac{\alpha.\gamma}{\beta.\delta}$ is constant.

56. [Desargues' Theorem.] ABCD is a quadrangle inscribed in a conic; a line cuts the conic at P, P′ and AD, BC, AC, BD, at Q, Q′, R, R′; prove that $\{PQRP'\}=\{PR'Q'P'\}=\{P'Q'R'P\}$.

57. ABC, PQR are two triangles inscribed in a conic; PQ, PR cut BC at Q', R'; AB, AC cut QR at B', C'; prove that

$$\{RB'CQ\}=\{R'BCQ'\}.$$

58. O, O' are points on the chord AA' of a conic; Q is another point on the conic; OPP' is a chord such that $Q\{AA'PP'\}=\{AA'OO'\}$, prove that $O'P'$ touches the conic.

CENTRAL PROPERTIES.

When a circle Σ is projected into a conic σ, the pole of the vanishing line w.r.t. Σ is projected into the pole of the line at infinity w.r.t. σ. This point enjoys a remarkable property, which will now be proved.

THEOREM 60.

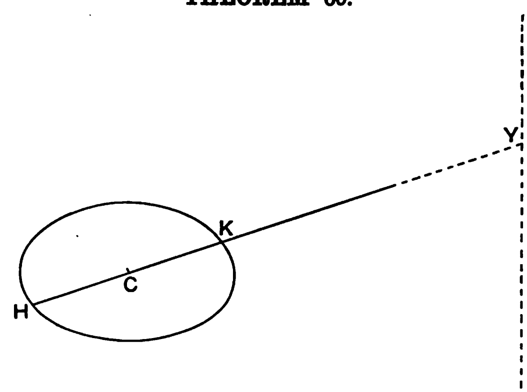

FIG. 45.

If C is the pole of the line at infinity w.r.t. a conic, then every chord through C is bisected at C.

[In this and following figures, ideal elements will be indicated by dots.]

Let HCK be any chord, meeting the line at infinity in the (ideal) point Y.

Then $\{HCKY\}$ is harmonic;

$\therefore\ HC = CK$. Q.E.D.

Definitions.

(1) The pole of the line at infinity w.r.t. a conic is called the **centre** of the conic.

(2) Any chord through the centre of a conic is called a **diameter**.

(3) The tangents to a conic from its centre are called the **asymptotes** of the conic.

(4) Conjugate lines through the centre of a conic are called **conjugate diameters**.

(5) A chord of a conic which is conjugate to a diameter is called a **double ordinate** to that diameter.

(6) If the asymptotes of a hyperbola are at right angles, it is called a **rectangular hyperbola**.

THEOREM 61.

The mid-points of a system of parallel chords of a conic lie on a diameter.

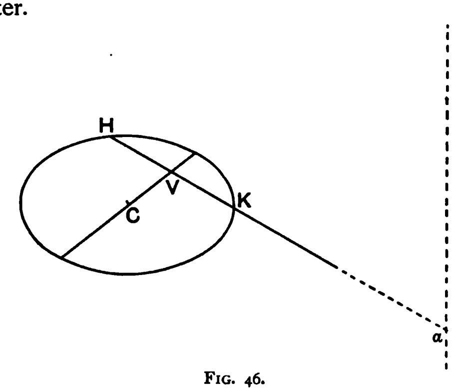

FIG. 46.

Let HK be any one of the system of parallel chords, which concur at the fixed ideal point α.

Let V be the mid point of HK.

Since $\{HVK\alpha\}$ is harmonic, the polar of α passes through V.

$\therefore\ V$ lies on a fixed line, viz. the polar of α, which is a diameter, since α is a point at infinity. Q.E.D.

THEOREM 62.

(1) If PCP', DCD' are two conjugate diameters of a conic, centre C, then PCP' bisects all chords parallel to DCD'.

(2) If HK is any double ordinate to PCP' meeting it at N, then HK is bisected at N, and is parallel to the tangents at P, P' to the conic, and to DCD'.

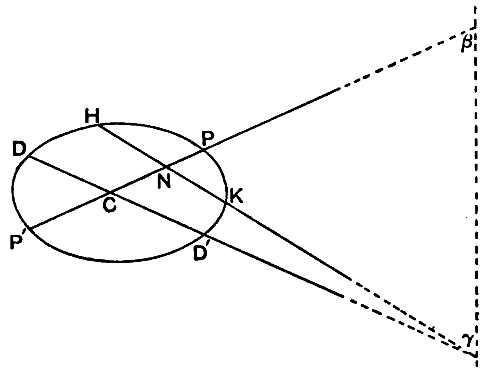

FIG. 47.

(1) PCP', DCD' meet the line at infinity at β, γ.

HK is any chord parallel to DCD' and therefore passes through γ.

Now $C\beta$ contains the pole C of $\beta\gamma$;

$\therefore$ $\beta\gamma$ contains the pole of $C\beta$.

But by hypothesis, $C\gamma$ contains the pole of $C\beta$;

$\therefore$ γ is the pole of $C\beta$.

$\therefore$ $\{HNK\gamma\}$ is harmonic;

$\therefore$ $HN = NK$. Q.E.D.

(2) With the notation of (1), since HK passes through the pole γ of $CP\beta$, it is conjugate to CP and is therefore the double ordinate from H to CP;

$\therefore$ the double ordinate from H to CP is bisected by CP, by (1).

And since γ is the pole of PCP', γP and $\gamma P'$ are the tangents at P, P'.

But the lines γP, $\gamma P'$, γCD, γKH concur at the ideal point γ and are therefore parallel. Q.E.D.

Definition.

If the chord HK is a double ordinate to the diameter PP' and meets it at N, then HN is called the **ordinate** to PP'.

59. Prove by projection that the centre of a conic is inside, on, or outside the curve according as the conic is an ellipse, parabola or hyperbola.

60. If two chords of a conic bisect each other, prove that each must be a diameter.

61. C is the centre of a conic; CP bisects one chord parallel to CD; prove that CP, CD are conjugate diameters.

62. T is the pole of a chord PQ of a conic; R is the mid-point of PQ; prove that TR passes through the centre of the conic.

63. The tangents at two points P, Q of a conic are parallel; prove that PQ is a diameter.

64. Prove that the cross ratio of the pencil formed by four diameters of a conic is equal to that of the pencil formed by the four conjugate diameters.

65. PP′ is a diameter of a conic, Q is any point on the curve; prove that QP, QP′ are parallel to a pair of conjugate diameters.

66. PCP′ is a diameter of a conic, centre C; QV is the ordinate from any point Q on the curve to PP′; the tangent at Q meets CP at T; prove that $\mathbf{CV \cdot CT = CP^2}$.

67. Given a point A and a circle Σ, prove that Σ can be projected into a conic having the projection of A as centre.

68. CA, CB are two perpendicular conjugate semi-diameters of a conic; T is the pole of a chord PQ; N is the foot of the perpendicular from T to CA; prove that PN, QN are equally inclined to TN.

69. QV is the ordinate from a point Q on a conic to a diameter PP'; a line through P parallel to the tangent at Q cuts QV, QP' at H, K; prove that $PH=HK$.

70. PP' is a diameter of a conic; the tangent at any point R meets the tangent at P in N; $P'R$ meets PN at Q; prove that $PN=NQ$.

71. p is the polar of P w.r.t. a conic, centre C; prove that the diameter conjugate to CP is parallel to p.

72. The normal QR at a point Q on a conic is an ordinate to the diameter PP'; prove that QR bisects $P\hat{Q}P'$.

73. QQ' is a double ordinate to the diameter PP' of a conic, centre C; R is any other point on the curve; RQ, RQ' meet PP' at L, M; prove that $CL \cdot CM = CP^2$.

74. Two variable diameters of a conic, which make equal angles with a fixed diameter PP', meet the tangent at P in K, K'; prove that the other tangents from K, K' to the conic intersect on a fixed line.

75. A conic touches the sides BC, CA, AB of a triangle at P, Q, R; A' is the mid-point of BC; prove that (1) the pole of AA' lies on a line through A parallel to BC, (2) AA', QR and the diameter through P are concurrent.

76. AB, AC are two chords of a conic; the diameter conjugate to AB meets AC at D; if P is the pole of BC, prove that PD is parallel to AB.

77. O is a fixed point on a fixed diameter PP' of a conic; a variable line through O cuts the conic at Q, Q'; PQ, PQ' meet the tangent at P' in R, R'; prove that $P'R . P'R'$ is constant.

78. T is the pole of a chord PQ of a conic; the line joining P to any point R cuts the conic at H; QH cuts the polar of R at K; prove that K lies on TR.

79. From a fixed point O is drawn a variable line, cutting a conic at P, Q; PN, QM are the ordinates to the diameter through O, prove that $\frac{1}{ON}+\frac{1}{OM}$ is constant.

80. P, Q are conjugate points w.r.t. a conic S; if the mid-point of PQ lies on S, prove that PQ is parallel to an asymptote of S.

81. PQ is a fixed diameter of a conic S; D is a fixed point on S. Two variable chords DH, DK cut PQ at H', K'; if $PH'=QK'$, prove that the locus of the pole of HK is a straight line.

THE PARABOLA.

If the vanishing line touches the generating circle, the projected conic touches the line at infinity and is then called a parabola; and the centre of the parabola, defined as the pole of the line at infinity, is a point at infinity, viz. the ideal point of contact of the parabola with the line at infinity. By taking account of the characteristic fact that the parabola touches the line at infinity, it is possible to deduce a group of theorems peculiar to it.

From Theorem 61, it follows that the mid-points of a system of parallel chords of a parabola lie on a straight line passing through the "centre" of the parabola, such a line being a diameter of the parabola. But the centre of the parabola is a point at infinity. Consequently all diameters of a parabola concur at a common point at infinity and are therefore parallel.

THEOREM 63.

T is the pole of a chord PQ of a parabola; the diameter through T meets the curve at V and PQ at N; then $TV=VN$ and $PN=NQ$.

Let the parabola touch the (ideal) line at infinity λ at the (ideal) point α; and let PQ meet λ at ϵ.

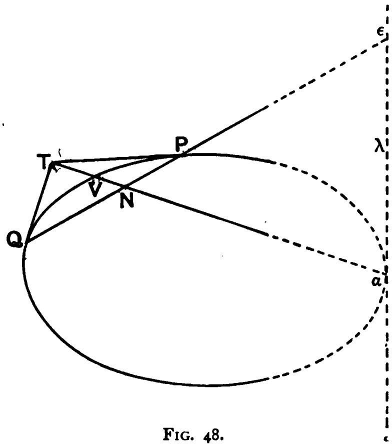

FIG. 48.

Since $TV\alpha$ meets the polar of T at N, $\{TN;\ V\alpha\}$ is harmonic.

$\therefore\ TV = VN.$ Q.E.D.

Since the polar of T passes through ϵ, the polar of ϵ passes through T;

$\therefore$ the polar of ϵ is $T\alpha$;

$\therefore\ \{PQ;\ N\epsilon\}$ is harmonic.

$\therefore\ PN = NQ.$ Q.E.D.

82. If PQ is a chord of a parabola, prove that the tangents at P, Q meet on the diameter bisecting PQ.

83. If PQ, $P'Q'$ are two parallel chords of a parabola, prove that PP', QQ' meet on the diameter bisecting PQ.

84. What property is obtained from Theorem 56 by taking $HCKD$ as the line at infinity?

85. What property is obtained from Theorem 56 by taking THA as the line at infinity?

86. Prove that a parabola cannot have two finite parallel tangents.

87. Deduce a property of the parabola from Brianchon's theorem: if a hexagon circumscribes a conic, the three lines joining pairs of opposite vertices are concurrent.

88. Deduce a property of the parabola from the theorem: ABC is a fixed triangle inscribed in a conic; P is a variable point on the conic; PL, PM, PN, PX are the perpendiculars from P to BC, CA, AB and the tangent at C, then $\dfrac{PL\,.\,PM}{PN\,.\,PX}$ is constant.

89. T is the pole of a chord PQ of a conic; R is the mid-point of PQ; TR meets the curve at V; if $TV=VR$, prove that the conic must be a parabola.

90. A, B, C are three fixed points on a parabola; P is a variable point on the curve; PB, PC cut the diameter through A at B', C'; prove that $\frac{AB'}{AC'}$ is constant.

91. T is the pole of a chord HK of a parabola; any diameter cuts TH, TK, HK at P, Q, R and the curve at V; prove that $\mathbf{VP \cdot VQ = VR^2}$. [Let a be the point at infinity on the curve, and note that $H\{HVKa\}=K\{HVKa\}$.]

92. T is the pole of a chord PQ of a parabola; any other tangent cuts TP, TQ at H, K; prove that HK is bisected by the tangent parallel to PQ. [Use Chasles' Theorem.]

93. T is the pole of a chord PQ of a parabola; any diameter cuts TP, TQ at P', Q' and a line through T parallel to PQ at R; prove that $P'R=RQ'$.

94. T is the pole of a chord PQ of a parabola; a line through P, parallel to TQ, meets the diameter through Q at R; TR meets PQ at H; prove that $PH=2HQ$.

[If V is the mid-point of PQ, prove that $\{PH;\ VQ\}$ is harmonic.]

95. PQ is a chord of a parabola, perpendicular to the diameter bisecting it; T is its pole; any tangent cuts TP, TQ at H, K; prove that $TH=KQ$. [Use Chasles' Theorem.]

96. ABC is a fixed triangle circumscribing a parabola; a variable tangent meets BC, CA, AB at X, Y, Z; prove that $\mathbf{\frac{XY}{YZ}}$ is constant. [Use Chasles' Theorem.]

97. ABC is a triangle circumscribing a parabola; BC is bisected at its point of contact; prove that any other tangent to the parabola is cut in two equal segments by the sides of the triangle.

98. A parabola is inscribed in the given triangle ABC, touching BC at the given point D; construct its point of contact with AB.

99. T is the pole of a chord PQ of a parabola; a diameter meets PQ in M, the curve in B, and a line through T parallel to the tangent at B in N; prove that $NB=BM$.

100. A variable tangent to a parabola cuts two fixed tangents at X, Y; prove that the locus of the mid-point of XY is a straight line.

101. PQ is a chord of a parabola; any diameter cuts PQ in N, the curve in B and the tangent at P in R; prove that $\frac{RB}{BN}=\frac{PN}{NQ}$.

THE HYPERBOLA.

Since the centre of a conic is the pole of the line at infinity, the asymptotes are, by definition, the lines joining the centre to the points of intersection of the curve with the line at infinity. Consequently an ellipse has two imaginary asymptotes, and a hyperbola has two real asymptotes. In the case of the parabola, the two asymptotes are each coincident with the line at infinity. The existence of the asymptotes is perceived most simply by regarding them as the projections of the tangents to the generating circle, at its points of intersection (imaginary, coincident, or real) with the vanishing line.

A number of elementary properties may be derived from this conception of the hyperbola.

THEOREM 64.

Any two conjugate diameters of a hyperbola are harmonically conjugate to the asymptotes; and, conversely, any two lines

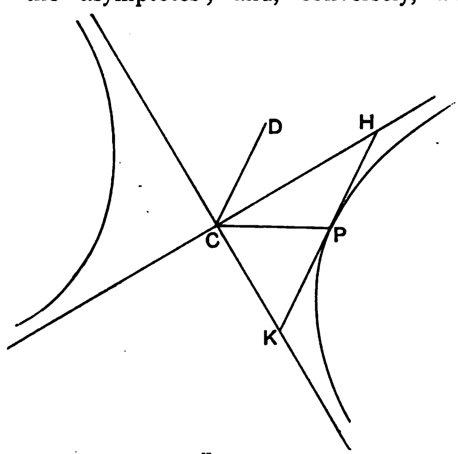

FIG. 49.

harmonically conjugate to the asymptotes are conjugate diameters.

This is simply a special case of Theorem 54.

THEOREM 65.

If a tangent at a point P of a hyperbola meets the asymptotes at H, K; then $HP = PK$.

This is simply a special case of Theorem 56.

THEOREM 66.

Conjugate diameters of a rectangular hyperbola are equally inclined to the asymptotes.

The proof is left to the reader. [Use Theorem 64.]

THEOREM 67.

If a variable tangent to a hyperbola, centre C, cuts the asymptotes at P, P'; then $CP.CP'$ is constant.

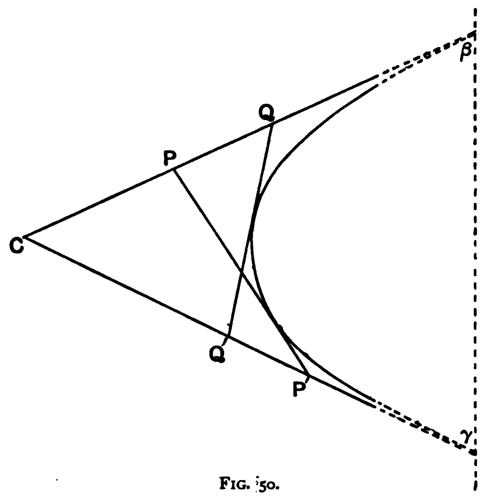

FIG. 50.

Let β, γ be the ideal points on the hyperbola, so that $C\beta$, $C\gamma$ are tangents. Draw any other tangent, cutting the asymptotes at Q, Q'.

Since by Chasles' theorem PP', QQ' and tangents very close to $C\beta$, $C\gamma$ cut off on $C\beta$, $C\gamma$ equicross ranges, in the limit we have

$$\{CP\beta Q\} = \{\gamma P'CQ'\};$$

$$\therefore \frac{CP.\beta Q}{CQ.\beta P} = \frac{\gamma P'.CQ'}{\gamma Q'.CP'}.$$

But
$$\frac{\beta Q}{\beta P} = 1 = \frac{\gamma P'}{\gamma Q'};$$

$$\therefore \frac{CP}{CQ} = \frac{CQ'}{CP'};$$

$$\therefore CP.CP' = CQ.CQ';$$

$$\therefore CP.CP' \text{ is constant.}$$

Q.E.D.

Corollary.

A variable tangent to a hyperbola makes with the asymptotes a triangle of constant area.

102. Prove Theorem 66.

103. Prove the Corollary of Theorem 67.

104. Prove that each asymptote of a hyperbola is a self-conjugate diameter.

105. T is the pole of a chord PQ of a conic; find a line through T meeting the curve at H and PQ at R, so that $TH=HR$. What is the condition that only one such line exists?

106. PP′ is a diameter of a rectangular hyperbola; Q is any point on the curve; prove that PQ, P′Q are equally inclined to the asymptotes.

107. T is the pole of a chord HK of a hyperbola; TH, TK, HK meet one asymptote at P, Q, R; prove that $PR=RQ$.

108. ABC are three fixed points on a hyperbola; P is a variable point on the curve; PB, PC meet the line through A parallel to one of the asymptotes at B', C'; prove that $\frac{AB'}{AC'}$ is constant.

109. A, B are two fixed points on a hyperbola; P is a variable point on the curve; PA, PB meet one asymptote at A', B'; prove that $A'B'$ is of constant length.

110. PP' is any diameter of a hyperbola; any chord $P'Q$ meets the lines through P parallel to the asymptotes at R, T; prove that $RQ=QT$.

111. A line through P parallel to an asymptote of a hyperbola cuts the curve at Q and the polar of P at R, prove that $PQ=QR$.

112. A straight line meets the asymptotes of a hyperbola in R, R' and a pair of conjugate diameters in K, K'; if O is the mid-point of RR', prove that $OR^2=OK.OK'$.

113. T is the pole of a chord MN of a hyperbola; any straight line parallel to one asymptote cuts TM, TN, MN at P, Q, R and the curve at B; prove that $BP.BQ=BR^2$.

114. P, Q are two points on a hyperbola, centre C; lines through P parallel to the asymptotes cut CQ at H, K; prove that $CH.CK=CQ^2$.

115. P, Q are two points on a hyperbola, whose asymptotes are $\mathbf{OP_1Q_1}$, $\mathbf{OP_2Q_2}$; if $\mathbf{OP_1PP_2}$, $\mathbf{OQ_1QQ_2}$ are parallelograms, prove that $\mathbf{OP_1 . OP_2 = OQ_1 . OQ_2}$.

[Let β, γ be the ideal points on the curve and note that

$$\beta\{\gamma P\beta Q\}=\gamma\{\gamma P\beta Q\}.]$$

116. Deduce Ex. 115 from Theorems 65, 67.

117. Prove that the equation of a hyperbola, referred to its asymptotes as axes, is of the form $xy=c^2$.

THE QUADRANGLE AND QUADRILATERAL.

THEOREM 68.

If a system of conics circumscribe a given quadrangle, the diagonal point triangle is a self-conjugate triangle w.r.t. each conic of the system.

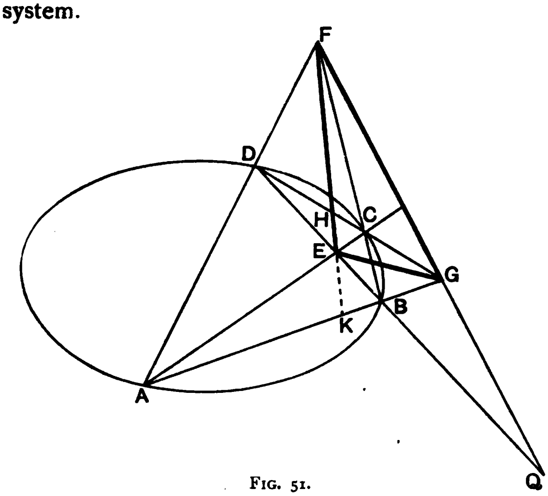

FIG. 51.

The proof is identical with that of Theorem 76, page 168, in Part I.

THEOREM 69.

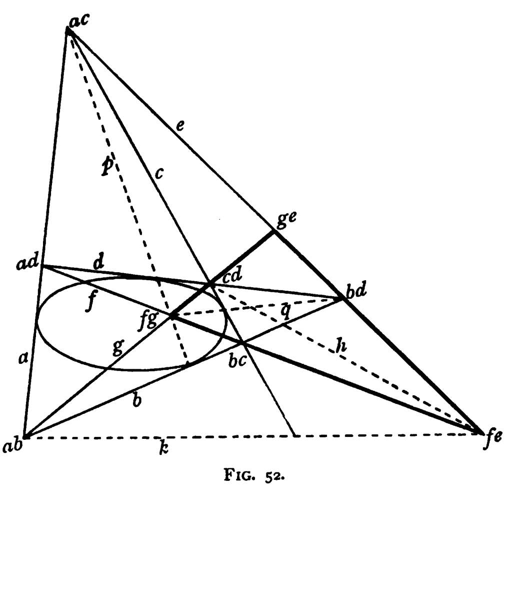

FIG. 52.

If a system of conics are inscribed in a given quadrilateral, the diagonal line triangle is a self-conjugate triangle w.r.t. each conic of the system.

The proof is identical with that of Theorem 77, page 169, in Part I.

THEOREM 70.

If a quadrangle is inscribed in a conic, and if a quadrilateral is formed by drawing the tangents at its vertices, then the diagonal

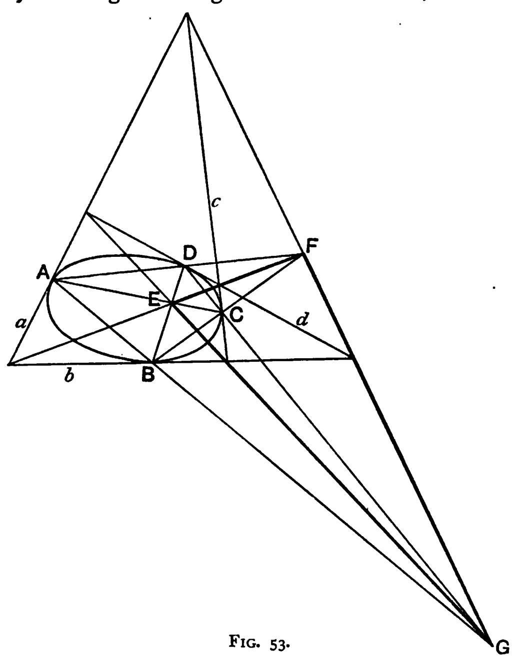

FIG. 53.

line triangle of the quadrilateral coincides with the diagonal point triangle of the quadrangle.

The proof is identical with that of Theorem 78, page 168, in Part I.

It is interesting to see what theorem, peculiar to the parabola, can be deduced from the general theory of the complete quadrilateral circumscribing a conic, by taking the line at infinity as one of the sides of the quadrilateral.

Let A, B, C, α be the points of contact of the four tangents.

Let the tangents at A, B, C form the triangle PQR, and let $P'Q'R'$ be the diagonal line triangle of the quadrilateral.

Let P, ϵ; Q, δ; R, γ be the pairs of opposite vertices of the quadrilateral.

Since $P'Q'$ meets PQ at γ, $P'Q'$ is parallel to PQ;

Similarly $Q'R'$, $R'P'$ are parallel to QR, RP.

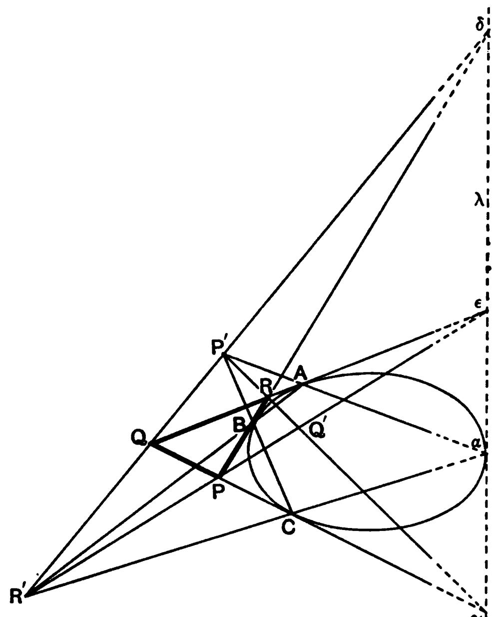

FIG. 54.

Since $P'A\alpha$; $Q'B\alpha$; $R'C\alpha$ are collinear sets of points, $P'A$, $Q'B$, $R'C$ are diameters of the parabola.

Hence we have the following theorem:

THEOREM 71.

Through the vertices of a triangle PQR, circumscribing a parabola, lines are drawn parallel to the opposite sides forming a triangle $P'Q'R'$; then $P'Q'R'$ is a self-conjugate triangle w.r.t. the parabola; and the diameters through P', Q', R' meet the curve at its points of contact with QR, RP, PQ.

118. Prove Theorem 68.

119. Prove Theorem 69.

120. Prove Theorem 70.

121. What theorem can be deduced from the fact that, in Fig. 54, A, Q', C and B, Q', α are collinear sets of points?

* THEOREM 72.

If a conic passes through a fixed point A, and has a fixed self-conjugate triangle PQR, then there are three other fixed points B, C, D, which lie on the conic, and PQR is the diagonal point triangle of the quadrangle $ABCD$. Further, every conic through A, B, C, D has PQR as a self-conjugate triangle.

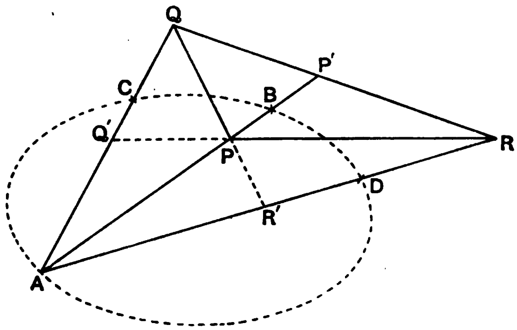

FIG. 55.

Join AP, AQ, AR cutting QR, RP, PQ at P', Q', R'.

Take points B, C, D dividing PP', QQ', RR' harmonically w.r.t. A.

Now P is the pole of QR; therefore since $\{PP';\ AB\}$ is harmonic, as the conic passes through A, it must pass through B.

Similarly the conic passes through C, D.

Since $\{APBP'\} = -1 = \{AQ'CQ\}$,

$\therefore$ PQ' BC, $P'Q$ are concurrent.

$\therefore$ BC passes through R.

Similarly BD passes through Q.

Since $\{AQ'CQ\} = -1 = \{ARDR'\}$,

$\therefore$ $Q'R$, CD, QR' are concurrent.

$\therefore$ CD passes through P.

$\therefore$ PQR is the diagonal point triangle of the quadrangle $ABCD$.

$\therefore$ by Theorem 68, every conic through A, B, C, D has PQR as a self-conjugate triangle. Q.E.D.

* THEOREM 73.

If a conic touches a fixed line a, and has a fixed self-conjugate triangle pqr, then there are three other fixed lines b, c, d, which touch the conic, and pqr is the diagonal line triangle of the

* At a first reading, these theorems may be omitted.

quadrilateral *abcd*. Further, every conic touching *a*, *b*, *c*, *d* has *pqr* as a self-conjugate triangle.

The proof is left to the reader.

[It is only necessary to apply the Principle of Duality to the method of proof of Theorem 72, see Part I., pages 168–9.]

122. Prove Theorem 73.

123. Prove that two conics, which do not have double contact, have one and only one common self-conjugate triangle.

124. $ABCD$ is a given quadrilateral; a variable conic touches AB, AD at B, D and cuts BC, DC again at H, K; prove that HK meets BD in a fixed point.

125. A variable conic passes through four fixed points; prove that the tangents at these points meet on fixed lines.

126. $ABCD$ is a quadrangle inscribed in a conic; AC, BD meet at P; any line through P cuts AB, CD at Q, R; prove that the perpendiculars from R, P, Q to the line half-way between P and its polar are in geometrical progression.

127. O is any point inside the triangle ABC; AO, BO, CO meet BC, CA, AB at D, E, F; prove that DEF is a self-conjugate triangle w.r.t. any conic through $ABCO$.

128. Two chords AB, CD of a conic intersect at O; P, Q are the poles of AD, BC; prove that P, O, Q are collinear.

129. What theorem is obtained from Ex. 128, by taking BC as the line at infinity?

130. Given a conic, show how to construct, with the use of a ruler only, the polar of a given point.

131. Given a conic, show how to construct, with the use of a ruler only, the pole of a given line.

132. Two tangents to a conic are fixed; two others are drawn so as to form with the fixed tangents a quadrangle, having two opposite sides along the fixed tangents; prove that one of the diagonal points lies on a fixed line.

133. Two variable chords PQ, RS of a given conic meet at a fixed point O; a fixed line through O cuts PS, QR at L, M; prove that $\frac{1}{OL}+\frac{1}{OM}$ is constant.

134. AA', BB' are two conjugate diameters of a conic; PQ is any chord; AP, AQ meet $A'Q$, $A'P$ at H, K; prove that HK is parallel to BB'.

135. O is the point of intersection of a pair of common chords of two conics S_1, S_2; a line through O cuts S_1 at P_1, Q_1 and S_2 at P_2, Q_2; prove that $\frac{1}{OP_1}+\frac{1}{OQ_1}=\frac{1}{OP_2}+\frac{1}{OQ_2}$.

136. T is the pole of a chord HK of a hyperbola, centre C; TH, TK meet one asymptote at Q, R and the other at Q', R'; prove that QR', $Q'R$, HK are parallel.

137. With the notation of Ex. 136, prove that CT bisects QR' and $Q'R$.

138. If a parallelogram is inscribed in a conic, prove that the diagonals intersect at the centre of the conic.

139. If a parallelogram is circumscribed about a conic, prove that the diagonals are conjugate diameters.

140. Straight lines are drawn parallel to the asymptotes of a hyperbola through the mid-point V of a chord PP' and meet the curve at Q, Q'. Prove that QQ' is parallel to PP'.

141. If two quadrangles have the same diagonal points, prove that their eight vertices lie either on a conic or four by four on two straight lines. [Use Ex. 123.]

142. A, B are two fixed points; PAQ is a variable chord of a given conic; BP, BQ meet the conic again at P', Q'; prove that $P'Q'$ passes through a fixed point.

143. Two conics are inscribed in the same quadrilateral, prove that their eight points of contact lie on a conic. [Use Ex. 141.] What is the dual theorem?

144. PQR is a self-conjugate triangle w.r.t. a conic; ZQX and XRY are chords; prove that YPZ is a straight line.

145. Given a conic and a point P outside it, construct the tangents from P to the conic, using a ruler only.

146. ABC is a triangle inscribed in a conic; prove that an unlimited number of self-conjugate triangles PQR can be drawn, such that P, Q, R lie on BC, CA, AB. Prove also that AP, BQ, CR are concurrent, and find the locus of their point of intersection.

147. P, Q, R are the mid-points of the sides of a triangle, self-conjugate w.r.t. a parabola; prove that the triangle PQR circumscribes the parabola.

148. G is the centroid of the triangle ABC; prove that the pole of the mid-point of BC w.r.t. any conic through A, B, C, G is the join of the mid-points of AB, AC.

The importance of the cross-ratio theory of the conic arises largely from the fact that the converse of Theorem 57 is true; which accordingly supplies a simple and fundamental descriptive test as to whether or not a given locus or envelope is a conic. A purely geometrical method of proof is not at all easy, and is therefore reserved to a later chapter (pages 251–3). By making use of results, obtained analytically in Chapter III., Theorem 51, it is, however, possible to give a rigorous proof of a very simple

character. The analysis of the dual theorem that one and only one conic can be drawn to touch five given lines, no four of which are concurrent, is somewhat longer, and was consequently omitted; it is, however, shown in Chapter VII. how a point equation can be determined from a given line equation; and by the aid of this piece of work, it is easy to complete the theorem (see also p. 259).

THEOREM 74.

A, B, C, D are four fixed points, no three of which are collinear; *P* is a variable point such that $P\{ABCD\}$ is constant; then the locus of *P* is a conic through *A, B, C, D*.

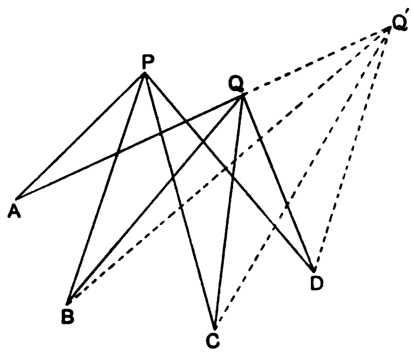

FIG. 56.

Let Q be any other position of P.

Draw a conic through P, A, B, C, D, and if it does not pass through Q, let its other meet with AQ be Q'.

Then $\quad Q\{ABCD\} = P\{ABCD\}$ by hypothesis

$\qquad\qquad\qquad\qquad = Q'\{ABCD\}$ by Chasles' theorem.

But $Q\{ABCD\}$, $Q'\{ABCD\}$ have a common corresponding ray.

$\therefore$ B, C, D are collinear, which is contrary to hypothesis.

$\therefore$ the locus of Q is the conic through P, A, B, C, D. Q.E.D.

This theorem may be stated in another form, which is more often useful in rider-work.

H, K are two fixed points; HP_1, HP_2, HP_3, ... and KP_1, KP_2, KP_3, ... are two pencils of lines through H, K meeting at P_1, P_2, P_3, ... If the cross ratio of **any four rays** of the first pencil is equal to the cross ratio of the four corresponding rays of the second pencil, then the points P_1, P_2, P_3, ... lie on a conic through H, K.

A simple analytical method is indicated in Ex. 149.

THEOREM 75.

a, *b*, *c*, *d* are four fixed lines, no three of which are concurrent; *p* is a variable line such that $p\{abcd\}$ is constant; then the envelope of *p* is a conic touching *a*, *b*, *c*, *d*.

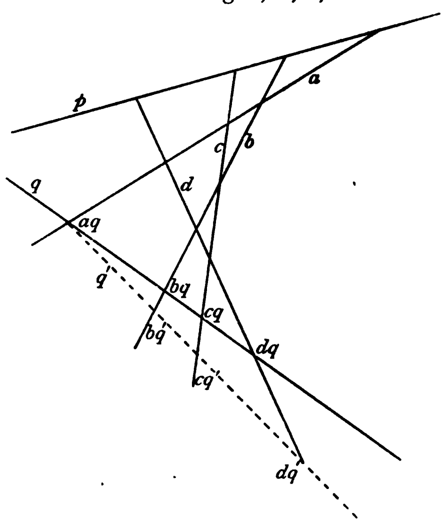

FIG. 57.

Let *q* be any other position of *p*.

Draw a conic touching *p*, *a*, *b*, *c*, *d*, and if it does not touch *q*, let the other tangent from *aq* be *q'*.

Then $\quad q\{abcd\} = p\{abcd\}$ by hypothesis

$\qquad\qquad = q'\{abcd\}$ by Chasles' theorem.

But $q\{abcd\}$, $q'\{abcd\}$ have a common corresponding point.

∴ *b*, *c*, *d* are concurrent, which is contrary to hypothesis.

∴ the envelope of *q* is the conic touching *p*, *a*, *b*, *c*, *d*.

Q.E.D.

This theorem may be stated in another form, which is more often useful in rider work.

h, *k* are two fixed lines; hp_1, hp_2, hp_3, ... and kp_1, kp_2, kp_3, ... are two ranges of points on *h*, *k* whose joins are p_1, p_2, p_3, ... If the cross ratio of **any four points** of the first range is equal to the cross ratio of the four corresponding points of the second range, then the lines p_1, p_2, p_3, ... envelope a conic touching *h*, *k*.

The following important property illustrates the application of the converse of Chasles' theorem. Further examples will be found in Chapter IX.

THEOREM 76.

A, B are two fixed points; P is a variable point such that PA, PB are conjugate lines w.r.t. a fixed conic; then the locus of P is a conic through A, B.

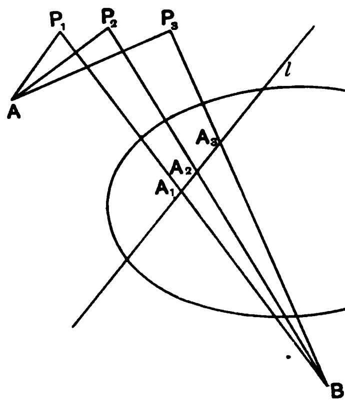

FIG. 58.

Draw any line AP_1 through A, and let A_1 be its pole. Join BA_1 and produce it to meet AP_1 at P_1; then AP_1, BP_1 are one pair of conjugate lines, for BP_1 contains the pole A_1 of AP_1. Similarly, construct any number of pairs of lines AP_2, BA_2P_2; AP_3, BA_3P_3; ....

Now, since the polar of A_1 passes through A, the polar of A passes through A_1.

$\therefore$ all the points A_1, A_2, A_3, ... lie on the polar l of A.

Since the cross ratio of any pencil of concurrent lines is equal to the cross ratio of the range formed by their poles,

$$\begin{aligned} A\{P_1P_2P_3\ldots\} &= \{A_1A_2A_3\ldots\} \\ &= B\{A_1A_2A_3\ldots\} \\ &= B\{P_1P_2P_3\ldots\}. \end{aligned}$$

Therefore, by the converse of Chasles' theorem, as stated in the second form, the locus of P is a conic through A, B.

Q.E.D.

By taking A, B as the circular points at infinity and using Theorems 11, 54, we have the following:

The locus of points, from which the tangents to a conic are at right angles, is a circle: called the **director circle** of the conic.

149. (1) Prove that the cross ratio of the four lines $u=0$, $u-\lambda v=0$, $v=0$, $u-\mu v=0$ is $\frac{\lambda}{\mu}$.

(2) If the equations of the lines HA, HB, HC are $u=0$, $u-\lambda v=0$, $v=0$, and the equations of $H'A$, $H'B$, $H'C$ are $u'=0$, $u'-\lambda' v'=0$, $v'=0$, and if HP, $H'P$ are variable lines such that $H\{ABCP\}=H'\{ABCP\}$, prove that the locus of P is the conic $\lambda' uv' - \lambda u'v=0$, which passes through H, H'.

150. If H{ABCD}=K{ABCH}, prove that the conic through H, K, A, B, C touches HD at H.

151. Enunciate and prove the dual theorem of Ex. 150.

152. The sides AB, AC of the triangle ABC are fixed in position, and $AB \backsim AC$ is constant; prove that BC envelopes a parabola.

153. A is the pole of a fixed chord BC of a given conic; a variable tangent to the conic cuts AB, AC at L, M; prove that BM, CL meet on a fixed conic, touching AB, AC at B, C.

154. ABC is a given triangle; P is a variable point on a fixed line; BP, CP meet AC, AB at Q, R; prove that QR touches a fixed conic. Does the conic touch AB, BC or CA?

155. A variable line L cuts the sides of a fixed triangle at X, Y, Z; if $\frac{\mathbf{XY}}{\mathbf{YZ}}$ is constant, prove that L envelopes a parabola.

156. The sides AB, AC of a triangle are fixed in position, and BC subtends a fixed angle at a given point; find the envelope of BC.

157. A variable tangent to a conic cuts two fixed tangents at P, Q; O is any fixed point; PP' is drawn parallel to OQ; prove that PP' envelopes a parabola.

158. P is a variable point on the base BC of a fixed triangle ABC; Q, R are the feet of the perpendiculars from P to AB, AC; prove that QR envelopes a parabola.

159. PQR is a variable triangle; PQ, PR pass through fixed points; QR touches a fixed conic, and Q, R lie on fixed tangents to the conic; find the locus of P.

160. A variable line, drawn from a fixed point, cuts the sides AB, AC of a given triangle ABC at P, Q; find the locus of the meet of BQ, CP.

161. A, B are two fixed points; PQ is a segment of constant length on a fixed line; prove that AP, BQ meet on a fixed hyperbola, and determine the directions of its asymptotes.

162. P is a variable point on a fixed line; PQ is a segment of a line passing through a fixed point, which subtends a constant angle at another fixed point; find the locus of Q.

163. The sides QR, RP, PQ of a variable triangle pass through three fixed points D, E, F; P lies on a fixed conic through E, F; Q lies on a fixed conic through F, D; prove that R lies on a fixed conic through D, E.

164. [Newton's Theorem.] POQ, PO′Q are two angles of constant magnitudes; O, O′ are fixed points; P moves on a fixed line; prove that Q traces out a conic through O, O′.

165. All the vertices, save one, of a polygon lie on fixed lines, and all the sides subtend given angles at given points; find the locus of the remaining vertex.

166. ABC is a given triangle; a variable line cuts AB, AC at X, Y, so that $BX=AY$; find the envelope of XY.

167. $ABCD$ is a trapezium of constant area; A, D are fixed points, and the parallel sides AB, CD are fixed in position; find the locus of the meet of AC, BD.

168. AB is a fixed chord of a given circle; PQ is a chord of given length; prove that AP, BQ meet on a fixed conic.

169. ABC, PQR are two triangles inscribed in a conic; prove that their six sides touch a conic.

[Let PQ, PR cut BC at Q', R' and AB, AC cut QR at B', C' and prove that $\{BQ'R'C\}=\{B'QRC'\}$.]

170. P is a variable point on a fixed line L; A, B are two fixed points; PQ is a diameter of the circle ABP; prove that the locus of Q is a hyperbola, whose asymptotes are perpendicular to AB and L.

171. ABC is a triangle inscribed in a conic; the tangents at the vertices meet the opposite sides in three points on a line L; if A moves on the conic, B, C being fixed, prove that L envelopes a conic.

172. A variable line passes through a fixed point; prove that the line, joining its poles w.r.t. two given conics, touches a fixed conic inscribed in the common self-conjugate triangle of the two conics.

173. P is the pole of a variable line L_1 through a fixed point A w.r.t. a given conic S_1; L_2 is the polar of P w.r.t. another given conic S_2; prove that L_1, L_2 meet on a fixed conic circumscribing the common self-conjugate triangle of S_1 and S_2.

174. A variable line moves so that its extremities lie on two fixed lines, and its mid-point on another fixed line; find its envelope.

175. AP is a variable chord of a given hyperbola. A is a fixed point; a line through A perpendicular to AP cuts a line through P parallel to an asymptote at P'; find the locus of P'.

176. From points on a fixed line, parallels are drawn to their polars w.r.t. a given conic; find their envelope.

177. The sides AB, AC of a triangle are fixed in position. If the circumcentre of ABC lies on a fixed line, prove that BC envelopes a parabola.

178. Given the base and the difference of the base angles of a triangle, find the locus of the vertex.

179. A variable line is drawn from a fixed point O to cut two fixed lines AB, AC at P, Q; prove that the locus of the mid-point of PQ is a hyperbola, having its asymptotes parallel to AB, AC.

180. A, B are fixed points on a conic; P is a variable point on the conic; Q is a point on PB such that $P\hat{A}Q$ is constant; find the locus of Q.

181. P is a variable point on a fixed line; the polar of P w.r.t. a given conic meets another given line at Q; find the envelope of PQ.

182. P, Q are conjugate points w.r.t. a given conic; PQ passes through a fixed point, and P lies on a fixed line; find the locus of Q.

183. What theorem is obtained from Ex. 182, by regarding the fixed line as the line at infinity?

184. P is a variable point on a fixed line; O is a fixed point; OP cuts a fixed conic at Q, R; H is a point such that $\{PH;\ QR\}$ is harmonic; prove that the locus of H is a conic.

185. The polars of a point P w.r.t. a system of coaxal circles concur at Q; L, L' are the limiting points; if P traces out a conic through L, L', prove that Q traces out another conic through L, L'.

186. Prove that in general two straight lines can be drawn, parallel to a given line, such that the segments intercepted on either by two other fixed lines subtend a given angle at a given point.

187. A variable circle passes through two fixed points A, B and cuts two fixed lines AC, AD at P, Q; find the envelope of PQ.

188. The sides AB, AC of a variable triangle are given in position and BC passes through a fixed point. Find the locus of a point dividing BC in a given ratio.

189. PQR is a self-conjugate triangle w.r.t. a given conic; P, Q lie on fixed lines, find the locus of R.

190. CP_1, CP_3; CP_2, CP_4 are two variable pairs of conjugate semi-diameters of an ellipse, the order of points on the curve being P_1, P_2, P_3, P_4; If $P_1P_2P_3P_4$ subtend a pencil of constant cross ratio at any point on the curve, prove that P_1P_2 envelopes a homothetic ellipse.

191. From a variable point H on the side AB of a fixed triangle ABC, lines HP, HQ are drawn to cut BC, AC at P, Q and make constant angles with AB; the circle HPQ cuts AB again at R. Prove that (1) the triangle PQR is of fixed shape; (2) PQ envelopes a parabola touching AC, BC; (3) the locus of the mid-point R' of PQ is a straight line L touching the same parabola; (4) the locus of the centroid of PQR is a straight line touching another parabola which touches AB, L and all positions of RR'.

192. If a variable triangle PQR of given shape is inscribed in a fixed triangle, prove that the locus of the mean centre of constant masses at P, Q, R is in general six straight lines. What interesting special cases are there of this theorem? [Use the method of Ex. 191.]

CHAPTER V.

GENERAL PROJECTION.

THE contents of this chapter summarise the application to the geometry of the conic of the theory of the circular points first developed systematically by Poncelet, in the light of his discovery of their connection with the foci. The existence of the real foci of a central conic, the name of which is due to Kepler, was known to Apollonius, who proved the characteristic property that the lines joining any point on the curve to the foci are equally inclined to the tangent at that point. The earliest mention of the focus of a parabola is found in the writings of Pappus, who enunciates the fundamental theorem that the distance of any point on the curve from the focus is proportional to its perpendicular distance from a fixed line (the directrix). Strange to say, however, this was passed over unnoticed, until attention was drawn to it by Newton in the *Principia*. An important advance was made by De Lahire, a pupil of Desargues, who saw that the directrix could be regarded as the polar of the focus, and the focus as a point through which conjugate lines are at right angles. These ideas were elaborated and amplified by the subsequent researches of Poncelet, Plücker, Steiner and Chasles.

The significance of the circular points at infinity has been explained in Chapter I. They were defined on an analytical basis, and were employed to give expression to a definite analytical phenomenon (see page 26). In view of this definition, their geometrical usage is intelligible only in so far as it connotes an analytical operation.

It is convenient to enumerate the properties which have already been established. The circular points at infinity will be denoted by ω, ω'.

(1) Every circle cuts the line at infinity at the same two points ω, ω' [page 25].

(2) Every conic, which passes through ω, ω', is a circle. [Theorem 20.]

(3) A system of concentric circles have double contact with each other at ω, ω'. [Theorem 18.]

(4) Any pair of lines at right angles are harmonically conjugate to the isotropic lines through their meet; and conversely, any two lines, which are harmonically conjugate to the isotropic lines, are at right angles. [Theorem 11.]

(5) If two concurrent pairs of perpendicular lines are each harmonically conjugate to another pair of lines, that pair must be the isotropic lines. [Theorem 12.]

(6) If a variable pair of lines include a constant angle, they form with the isotropic lines through their meet a pencil of constant cross ratio; and conversely, if two variable lines form with the isotropic lines a pencil of constant cross ratio, then they include a constant angle. [Theorem 11.]

THEOREM 77.

If C is the centre of a circle, $C\omega$ and $C\omega'$ are the asymptotes of the circle.

C is the pole of the line at infinity.

$\therefore$ $C\omega$, $C\omega'$ are the tangents to the circle at ω, ω'.

$\therefore$ by definition, $C\omega$, $C\omega'$ are the asymptotes of the circle.

Q.E.D.

It should be noted that this supplies another proof of (3).

For every circle centre C touches $C\omega$, $C\omega'$ at ω, ω'; and therefore all circles, having C as centre, touch each other at ω, ω'.

THEOREM 78.

If a conic has two pairs of perpendicular conjugate diameters, it must be a circle.

The asymptotes of the conic are harmonically conjugate to each pair of conjugate diameters. [Theorem 64.]

$\therefore$ the asymptotes must be the isotropic lines, by (5);

$\therefore$ the conic passes through ω, ω';

$\therefore$ the conic is a circle. Q.E.D

By assuming that two conics cannot cut at more than four points, it is possible to give another proof of this theorem.

Let C be the centre of the conic, and P any point on the conic.

Let PP_1, PP_2, PP_3, PP_4 be the double ordinates to the two pairs of perpendicular conjugate diameters. Then, since these ordinates are perpendicular to the diameters, we have, by congruent triangles, $CP = CP_1 = CP_2 = CP_3 = CP_4$;

$\therefore$ the circle centre C, radius CP, cuts the conic at five points;

$\therefore$ the conic must coincide with the circle. Q.E.D.

THEOREM 79.

(1) Any conic can be projected into a circle having the projection of any given point (not on the circle) as centre.

(2) Any conic can be projected into a circle, and at the same time any given line (not touching the circle) into the line at infinity.

(3) Any two points can be projected into ω, ω'.

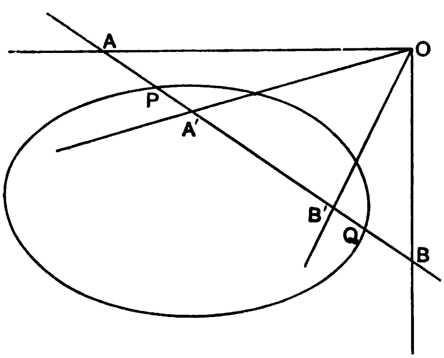

FIG. 59.

(1) Let O be the given point, and AB its polar w.r.t. the conic.

Draw any two pairs of conjugate lines OA, OA'; OB, OB' through O.

Project the angles AOA', BOB' into right angles and the line AB to infinity.

O is therefore projected into the centre of the new conic.

Therefore OA, OA'; OB, OB' are projected into two pairs of perpendicular conjugate diameters.

Therefore, by Theorem 78, the new conic is a circle, having the projection of O as centre. Q.E.D.

(2) Let AB be the given line; then using the process of (1), we obtain the required result.

(3) Let AB be the line passing through the given points P, Q.

Draw any conic through P, Q, and project the conic into a circle and the line AB to infinity.

Then P, Q project into the meets of a circle with the line at infinity, *i.e.* ω, ω'. Q.E.D.

THEOREM 80.

(1) If a system of conics have two common points, they can be projected into a system of circles.

(2) If a system of conics have four common points, they can be projected into a system of coaxal circles.

(3) If a system of conics have double contact with each other at the same two points, they can be projected into a system of concentric circles.

The proof is left to the reader.

Corollary.

Any two conics can be projected into two circles.

[Any two conics have four common points, real or imaginary.]

In order to acquire ease in applying the method of general projection, it is best first of all to practise the reverse process. Starting with a known property of the circle, we proceed to deduce from it the analogous generalised property of the conic. The following example will serve as an illustration.

EXAMPLE.

Generalise by projection the theorem: if two circles cut ortho-

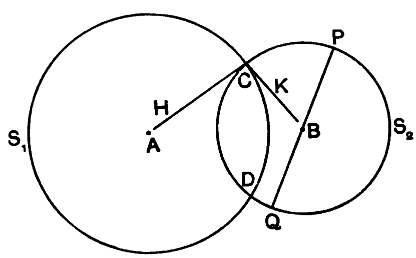

FIG. 60.

gonally, the extremities of any diameter of one of the circles are conjugate points w.r.t. the other.

Let A, B be the centres of the two circles S_1, S_2, which intersect orthogonally at C, D; let CH, CK be the tangents at C.

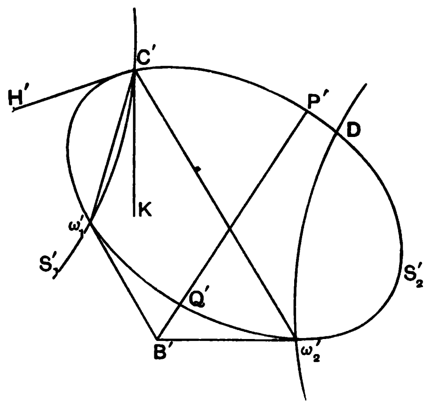

FIG. 61.

Let PBQ be any diameter of S_2.

We shall denote by dashes corresponding points in the generalised figure.

Now S_1, S_2, intersect at C, D and the circular points ω_1, ω_2, and may therefore be projected into two conics S_1', S_2' cutting at C', D', ω_1', ω_2'.

CH, CK project into the tangents $C'H'$, $C'K'$ to S_2', S_1'.

Since $H\hat{C}K = 90°$, $C\{HK;\ \omega_1\omega_2\}$ is harmonic;

therefore $C'\{H', K';\ \omega_1'\omega_2'\}$ is also harmonic.

B is the pole of $\omega_1\omega_2$ (the line at infinity) w.r.t. S_2, therefore B' is the pole of $\omega_1'\omega_2'$ w.r.t. S_2'. P, Q become the points of intersection P', Q' of any line through B' with S_2' and P', Q' are conjugate points w.r.t. S_1'.

Hence the theorem (omitting the dashes):

Two conics S_1, S_2 intersect at C, D, ω_1, ω_2; if the tangents at C to S_1, S_2 are harmonically conjugate to $C\omega_1$, $C\omega_2$, and if B is the pole of $\omega_1\omega_2$ w.r.t. S_2, then any line through B is cut by S_2 in two points which are conjugate w.r.t. S_1.

1. Prove Theorem 80.

2. What conic-property can be deduced from: if two circles touch at A and if any line through A meets them again at P, Q, then the tangents at P, Q are parallel.

3. Generalise by projection: if two circles touch each other, the line joining their centres passes through the point of contact.

4. Generalise by projection: if two circles intersect at A, B, the line joining their centres bisects AB at right angles.

5. Generalise by projection: if each of three circles touches the other two, then the tangents at their points of contact are concurrent.

6. Generalise by projection: if two circles intersect at A, B, the angle between the tangents at A is equal to the angle between the tangents at B.

7. Generalise by projection: angles in the same segment of a circle are equal.

8. Generalise by projection: the angle in a semi-circle is a right angle.

9. Generalise by projection: if a variable circle touches externally two fixed circles at P, Q, then the tangents at P, Q to the fixed circles meet on a line, viz. the radical axis of the fixed circles.

10. Generalise by projection: with the notation of Ex. 9, PQ passes through a fixed point, viz. a centre of similitude of the fixed circles.

11. Generalise by projection: if a variable circle touches two fixed circles, the locus of its centre is two conics.

12. Generalise by projection: if the tangents from a variable point P to a fixed circle include a constant angle, the locus of P is a concentric circle.

13. If three conics have one common chord, prove that their other common chords are concurrent.

14. Two conics have double contact; prove that a chord of one which touches the other is divided harmonically by its point of contact and the common chord.

15. Generalise by projection: if PQ, RS are parallel chords of concentric circles, then P, Q, R, S are concyclic.

16. Generalise by projection: if L, M, N are points on the sides BC, CA, AB of a triangle, then the circles AMN, BNL, CLM, have one common point.

17. Generalise by projection: ABC is a fixed triangle; if a variable circle passes through A and has B, C as conjugate points, then the locus of its centre is a straight line.

18. $\{ABCD\}$ is a harmonic range; O is any point outside AD; two conics are inscribed in the triangles OBC, OAB, respectively, and intersect at two points on OD; prove that two of their common tangents meet at D.

19. BC is a fixed chord of a given conic S; a variable line through a fixed point A cuts S at P, P' and BC at Q; Q' is the harmonic conjugate of Q w.r.t. P, P'; prove that the locus of Q' is a conic through A, B, C.

20. A variable conic passes through two fixed points A, B and touches two fixed lines; prove that the locus of the pole of AB is two straight lines.

21. Prove that each pair of common chords of two conics meets a common tangent in points which are harmonic conjugates w.r.t. the points of contact.

22. Three fixed conics have double contact with each other at the same two points; prove that any line touching one of them is cut by the others in a constant cross ratio.

23. Prove that a system of conics circumscribing a given triangle can be transformed by projection and inversion into a system of straight lines.

24. A, B are two fixed points on a fixed conic; P is a variable point on a fixed straight line; PA, PB meet the conic again at R, S; find the envelope of RS.

25. Two triangles are in perspective, and their six vertices lie on a conic S; prove that the pole of the axis of perspective w.r.t. S is the centre of perspective.

26. From a fixed point O on a conic, chords OP, OP′ are drawn equally inclined to a fixed chord OA; prove that PP′ passes through a fixed point. [If AB is the chord which subtends a right angle at O, project A, B into ω, ω'.]

27. A polygon of $2n$ sides is inscribed in a conic; if the joins of its opposite vertices concur at a point G, prove that the joins of the opposite vertices of the polygon, formed by the tangents to the conic at the vertices of the first polygon, also concur at G.

28. Generalise by projection: if the mid-point of a variable chord PQ of a circle lies on a fixed line, then PQ envelopes a parabola.

29. A triangle is inscribed in a conic S; two of its sides pass through fixed points A, B; if AB meets S at C, D, prove that the third side envelopes a conic, having double contact with S at C, D.

30. Generalise by projection: a variable circle passes through a fixed point and touches a fixed line, then the envelope of the polar of another fixed point is a conic.

State also the dual of the generalised theorem.

31. Generalise so as to obtain a theorem for two parabolas: it is possible to draw two circles to touch a given circle S at a given point, and to touch a given line l which intersects S; moreover the points of contact with l are harmonically conjugate to the meets of l and S.

32. Generalise by projection: two circles touch at A; PR, QS are two chords such that A, P, Q and A, R, S are collinear, then PR, QS are parallel.

33. Two conics have double contact at A, B; prove that the polars of any given point w.r.t. the conics meet on AB.

34. A variable conic passes through four fixed points, prove that the meets of its tangents at these points lie on three fixed lines.

35. ABC is a triangle inscribed in a conic; prove that the tangents at the vertices meet the opposite sides in three collinear points. [If the three points are P, Q, R, project the conic into a circle and PQ to infinity.]

36. H, K are the poles of two chords PQ, RS of a conic; prove that H, K, P, Q, R, S lie on a conic.

37. Two conics touch at P and cut at Q, R; the tangent at P meets QR at T; a line through T cuts one conic at L, M; PL, PM cut the other conic at L', M'; prove that $L'M'$ passes through T.

38. Generalise by projection: a line cuts two concentric circles, the first at A, B and the second at C, D; then AB, CD have the same mid-point.

39. Generalise by projection: if a variable circle touches a given circle and cuts a given line at a given angle, then it touches a second fixed circle: and the line is the radical axis of the two fixed circles.

40. A variable conic touches two fixed lines and passes through two fixed points A, B; prove that the chord of contact passes through one of two fixed points H, K; and that $\{AB; HK\}$ is harmonic.

41. A system of conics pass through four fixed points A, B, C, D; tangents are drawn from a fixed point on AB to the system; prove that the locus of the points of contact is a conic through C, D. Write down the dual theorem.

42. C is the centre of a conic; if Cω, Cω' are conjugate lines, prove that the conic is a rectangular hyperbola.

43. Three conics pass through four fixed points; prove that a common tangent to two of them is divided harmonically by the third.

44. P, Q are two fixed points on a fixed conic σ; a variable conic S passes through P, Q and two other fixed points, and meets σ at H, K; prove that HK passes through a fixed point.

45. Through a fixed point D, a variable conic is drawn having double contact with a fixed conic S; prove that the chord of contact meets the tangent at D on a fixed line, viz. the polar of D w.r.t. S.

46. AB is a common chord of two conics; PQ, RS are chords of the two conics; if P, R, A and Q, S, B are collinear sets of points, prove that PQ, RS meet on the other common chord.

47. Prove by projection the harmonic properties of a conic inscribed in a quadrilateral and circumscribing a quadrangle. [Project the inscribed quadrangle into a parallelogram and the conic into a circle, and note that a cyclic parallelogram must be a rectangle.]

48. [Brianchon's theorem.] If a hexagon circumscribes a conic, prove that the joins of opposite vertices are concurrent. [If two of the joins meet at O, project the conic into a circle, having the projection of O as centre.]

49. PAQ, PBR are chords of a fixed circle PQR; A, B are fixed points; RS is a chord parallel to AB; prove that QS passes through a fixed point.

50. Two conics touch at A; P is a variable point on the common tangent at A; Q, R are the points of contact of the tangents from P to the conics; prove that QR passes through a fixed point.

51. The chords PP', QQ', RR', SS', TT', WW' of a conic are concurrent; if a conic can be drawn to touch PQ, QR, RS, ST, TW, WP, prove that a conic can also be drawn to touch $P'Q'$, $Q'R'$, $R'S'$, $S'T'$, $T'W'$. $W'P'$.

52. PQR is a triangle inscribed in a fixed conic; PQ, PR are fixed in direction; prove that QR envelopes a conic having the same asymptotes as the given conic.

53. The sides PQ, PR of a triangle self-conjugate w.r.t. a given conic cut any tangent to the conic at H, K; H', K' are harmonic conjugates of H, K w.r.t. PQ, PR respectively.

Prove that $H'K'$ touches the conic.

Deduce a special theorem for the parabola.

Write down the dual theorem.

54. Prove that the locus of the centres of hyperbolas passing through two fixed points and having their asymptotes parallel to two given lines is a straight line.

55. A is a fixed point; P is a variable point on a fixed line; AP cuts a fixed conic at Q, R; find the locus of the harmonic conjugate of P w.r.t. Q, R.

56. T is a pole of a chord PQ of a conic S_1; a conic S_2 is drawn having TP, TQ as asymptotes; prove that one pair of common chords of S_1 and S_2 is parallel to PQ.

57. Two conics S_1, S_2 cut at A, B, C, D; if H is the pole of AB and CD w.r.t. S_1 and S_2 respectively, prove that there exists a point K which is the pole of CD, AB w.r.t. S_1, S_2, and that the six points A, B, C, D, H, K lie on a conic.

58. A variable conic touches two fixed lines at fixed points; PQ is a chord of the conic, the pole of which is fixed; find the locus of P and Q.

59. PQR is a triangle, self-conjugate w.r.t. a conic S; prove that an unlimited number of triangles can be inscribed in S, which are circumscribed to PQR.

60. A variable conic passes through four fixed points A, B, C, D and cuts two fixed lines AE, BF at H, K; prove that HK meets CD at a fixed point.

61. **[Pascal's theorem.] The three pairs of opposite sides of a hexagon, inscribed in a conic, meet at P, Q, R; prove that P, Q, R are collinear.** [Project the conic into a circle and PQ to infinity.]

62. Two common tangents of two conics meet at A; any two secants APQ, ARS cut off on the two conics the chords PR, QS; prove that PR, QS meet on one of the common chords of the two conics, provided the proper position, of the two possible positions, of S is chosen.

63. A variable conic passes through three fixed points and has two fixed points as conjugate points; prove that it passes through a fourth fixed point.

64. If a system of conics, circumscribing a fixed quadrangle, are projected into a system of coaxal circles, prove that two of the diagonal points project into the limiting points.

65. Two conics are inscribed in the triangle ABC; PQ is one of their common chords; the fourth common tangent meets BC at D; AF is the harmonic conjugate of AD w.r.t. AB, AC; prove that $A\{FD; PQ\}$ is harmonic.

66. Two conics touch the same line at P, Q and cut at A, B, C, D; prove that a conic through A, B, C, D and the mid-point of PQ has one asymptote parallel to PQ.

67. A tangent to a conic meets two conjugate lines AB, AC w.r.t. the conic, at B, C; the other tangents from B, C to the conic meet at D; prove that A, D are conjugate points w.r.t. the conic.

68. Two parabolas touch at P and cut at Q, R; prove that PQ, PR are harmonically conjugate to the diameters through P of the two curves.

69. PQ, PQ' are two chords of a conic equally inclined to the normal chord PP'; prove by projecting Q, Q' into ω, ω', that $P'Q$, $P'Q'$ are harmonically conjugate w.r.t. $P'P$ and the tangent at P'.

70. Prove that a triangle inscribed in a conic can be projected into an equilateral triangle inscribed in a circle.

71. A, B are two fixed points; P is a variable point such that PA, PB are conjugate lines w.r.t. a fixed conic σ; prove that the locus of P is a conic through A, B. [Project σ into a circle having the projection of A as centre.]

Deduce, by taking A, B as the circular points, that the locus of a point from which the tangents to a conic are at right angles is a circle.

72. A polygon is inscribed in a conic S_1 and circumscribes a conic S_2; the lines joining its vertices to any given point cut S_1 at the vertices of a new polygon; prove that the sides of this polygon also touch a conic.

73. T is the pole of a fixed chord PQ of a conic S; a variable tangent cuts TP, TQ at H, K; prove that the locus of the mid-point of HK is a conic, having double contact with S.

74. Generalise by projection: a straight line cuts two circles at A_1, A_2 and B_1, B_2 and their radical axis at O, then $\dfrac{OA_1}{OB_1}=\dfrac{OB_2}{OA_2}$.

75. $ABCD$ is a given quadrilateral; E is a fixed point on CD; P is a variable point such that $P\{ACBE\}=P\{AEBD\}$; prove that the locus of P is a conic circumscribing the triangle ABE, and having C, D as conjugate points.

THE FOCI.

Definition.

(1) If a quadrilateral is formed by drawing the pairs of tangents from the circular points at infinity, ω, ω' to a conic, and if S, H; S', H' are the other two pairs of opposite vertices, then S, H and S', H' are called the **foci** of the conic.

(2) Further, the lines SH, $S'H'$ are called the **principal axes** of the conic.

(3) The polars of the foci w.r.t. the conic are called the **directrices** of the conic; and the chord through a focus parallel to the corresponding directrix is called the **latus rectum.**

(4) If a system of conics have the same foci, they are said to be **confocal.**

THEOREM 81.

(1) Every real conic has four foci, of which two are real and two are conjugate imaginaries.

(2) Every real conic has two real principal axes, which are a pair of perpendicular conjugate diameters of the conic.

(3) Every pair of conjugate lines through any focus of a conic are at right angles; and conversely two perpendicular lines through a focus are conjugate lines.

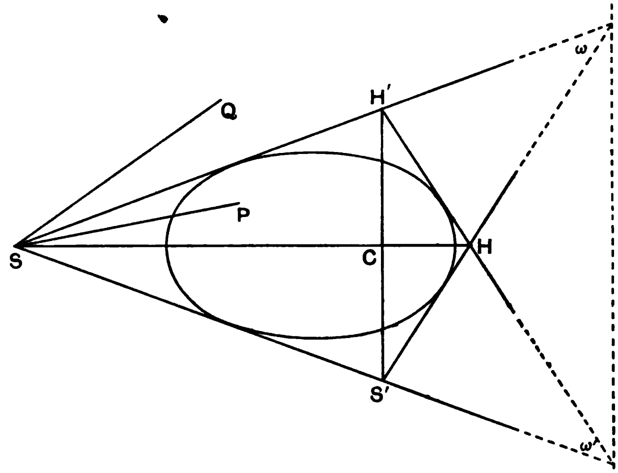

FIG. 62.

(1) ω, ω' are conjugate imaginaries.

Therefore the pair of tangents ωS, ωH are the conjugate imaginaries of the pair of tangents $\omega' S$, $\omega' H$, since the conic is real.

But two conjugate imaginary lines meet at a real point. [Th. 1.]

Therefore either ωS, $\omega' S$; ωH, $\omega' H$ *or* ωS, $\omega' H$; ωH, $\omega' S$ meet at real points, but not all four pairs, since the lines SS', etc., are imaginary.

Suppose, then, that the first pair meet at real points;

$\therefore$ S, H are real.

In that case, ωS, $\omega' H$ are the conjugate imaginaries of ωH, $\omega' S$. Therefore their meets H', S' are conjugate imaginary points.

Q.E.D.

(2) Since S, H are real and S', H' are conjugate imaginaries, the lines SH, $S'H'$ are both real. [Th. 2.]

From the theory of the circumscribed quadrilateral, the meet C of SH, $S'H'$ is the pole of $\omega\omega'$, *i.e.* the line at infinity, and is therefore the centre of the conic.

Also CH, CH' are conjugate lines w.r.t. the conic. [Th. 69.]

Therefore CH, CH' are conjugate diameters.

But $C\{HH'; \omega\omega'\}$ is harmonic, therefore $H\hat{C}H' = 90°$.

Therefore the two principal axes are a real pair of perpendicular conjugate diameters. Q.E.D.

(3) If SP, SQ are a pair of conjugate lines through S,

$S\{PQ; \omega\omega'\}$ is harmonic, [Th. 54]; therefore $P\hat{S}Q = 90°$. Q.E.D.

Conversely, if $P\hat{S}Q = 90°$, $S\{PQ; \omega\omega'\}$ is harmonic;
and therefore SP, SQ are conjugate lines. [Th. 54.] Q.E.D.

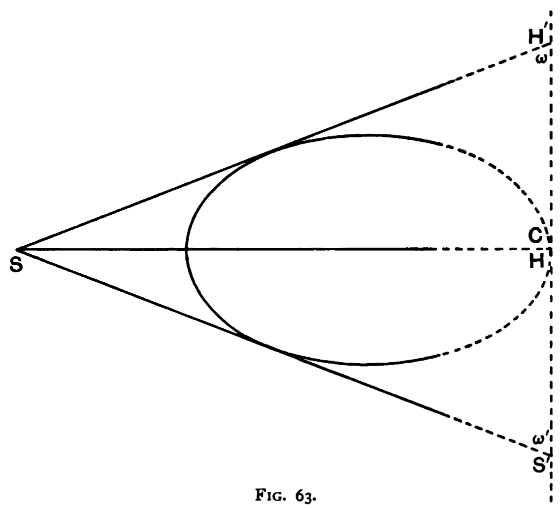

FIG. 63.

Corollary 1.

The four directrices form a parallelogram whose sides are parallel to the principal axes.

The proof is left to the reader.

Corollary 2.

A conic is a curve which is symmetrical about each of two lines at right angles, viz. the two principal axes.

This fundamental theorem is modified slightly, if the conic is a parabola [see Fig. 63].

In this case, by hypothesis, the line $\omega\omega'$ touches the conic, at H say. The lines $\omega' HH'$, $\omega HS'$, $H'CS'$ of Fig. 62, then coincide with $\omega\omega'$; the points H', S' coincide with ω, ω'; and C coincides with H.

A parabola therefore has one focus at a finite point, one focus at its centre, which is the point at infinity on the curve, and the other two foci at the circular points. It has one finitely situated principal axis, which bisects all chords perpendicular to it; while the other principal axis is the line at infinity.

If the conic is a circle, a further interesting special case arises, which the reader should examine for himself (see Ex. 79).

It is important that the reader should understand that Figs. 62, 63 have no purely geometrical counterpart. As has been said already, the circular points at infinity (as previously defined) have no meaning apart from analysis. This method of establishing the existence of the foci of a conic is simply a statement of an analytical process, which can be expressed more briefly, and is more easily apprehended, if geometrical terms are used; while the figures merely serve the purpose of indicating the substance of the arguments employed.

THEOREM 82. [PAPPUS' THEOREM.]

S is the focus of a conic; P is a variable point on the curve; M is the foot of the perpendicular from P to the directrix, corresponding to S; then $\dfrac{SP}{PM}$ is constant.

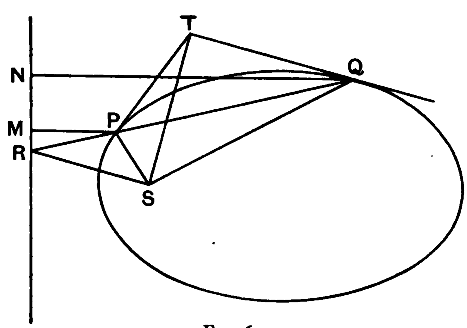

FIG. 64.

Let N be the foot of the perpendicular from any other point Q on the curve to the directrix.

Let PQ meet the directrix in R, and let T be the pole of PQ.

Now the polar of S (viz. the directrix) passes through R;

$\therefore$ the polar of R passes through S.

Also the polar of T passes through R; therefore the polar of R passes through T;

$\therefore$ the polar of R is TS;

$\therefore$ ST, SR are conjugate lines;

$\therefore$ $R\hat{S}T = 90°$. [Theorem 81 (3).]

But $S\{RT; PQ\}$ is harmonic, since ST is the polar of R;

$\therefore$ SR, ST are the bisectors of $P\hat{S}Q$;

$$\therefore \quad \frac{SP}{SQ} = \frac{PR}{QR}$$

$$= \frac{PM}{QN}, \text{ by parallels;}$$

$$\therefore \quad \frac{SP}{PM} = \frac{SQ}{QN};$$

$$\therefore \quad \frac{SP}{PM} \text{ is constant.}$$

Q.E.D.

Corollary.

If the conic is a parabola, $\dfrac{SP}{PM} = 1$.

Let the diameter through S cut the directrix at X and the curve at A, α, [α being at infinity].

SA is perpendicular to the directrix by Theorem 81, Cor. 1.

Further $\{A\alpha; SX\}$ is harmonic, so that $\dfrac{SA}{AX} = 1$.

76. Prove Theorem 81, Cor 1.

77. Prove that the centre of a conic bisects the line joining the two foci on either principal axis.

78. Prove that all parabolas, having a common focus, may be regarded as inscribed in a common triangle.

79. Prove that the four foci of a circle coincide with its centre. Where are the directrices? Draw a figure, analogous to Fig. 62.

80. Prove that the finite focus of a real parabola is a real point.

81. Prove that any conic through the four foci of a conic must be a rectangular hyperbola.

82. It is said that the four foci of a conic form a triangle and its orthocentre. Discuss the validity of this statement.

83. The centre S of a circle is a focus of a conic; prove that S may be regarded as a meet of two of the common tangents of the circle and conic.

84. S_1, S_2 are two conics inscribed in a quadrilateral $ABCD$; S_3 is any conic through A, B, C, D. If S_1, S_2 are projected into confocal conics, prove that S_3 will become either a circle or a rectangular hyperbola.

85. A side BC of a triangle ABC, self conjugate w.r.t. a conic, meets a directrix at P; if S is the corresponding focus, prove that $P\hat{S}A = 90°$.

86. $ABCD$ is a quadrilateral circumscribing a conic; if the diagonals AC, BD meet at a focus, prove that they are at right angles, and that the third diagonal is a directrix.

87. Prove that a conic, and two points off it, can be projected into a conic and its foci.

88. Prove that any two conics can be projected into confocal conics.

89. Prove that a conic, and two points off it, can be projected into a conic, its centre and one focus.

90. Given a point A on a conic, and a point B off it, prove that the conic can be projected into a parabola having the projections of A, B as vertex and focus.

91. If the tangents at two points P, Q of a conic, focus S, meet at T, prove that TP, TQ subtend equal or supplementary angles at S.

92. PQ is a variable chord of an ellipse, through a focus S; H is the other real focus; the bisector of $P\hat{H}Q$ meets PQ at R; prove that the locus of R is a conic through S, H. [Use Ex. 91.]

93. T is the pole of a chord PQ of a conic; N is the foot of the perpendicular from T to a principal axis; prove that NP, NQ are equally inclined to the axis.

94. P is a point on a conic, centre C, focus S; CP meets the directrix, corresponding to S, at K; prove that the tangent at P is perpendicular to SK.

95. A chord PQ, passing through the focus S of a conic, meets the corresponding directrix at R; prove that $\frac{1}{RP}+\frac{1}{RQ}=\frac{2}{RS}$. Hence prove that $\frac{1}{SP}+\frac{1}{SQ}$ remains constant as PQ turns about S. [Use Theorem 82.]

96. The tangent at P to a conic meets a directrix at R; prove that PR subtends a right angle at the corresponding focus.

97. The directrix of a hyperbola meets an asymptote CT at T; S is the corresponding focus; the other tangent from T to the curve touches it at P; prove that SP is parallel to CT.

98. PSP', QSQ' are two focal chords of a conic; QQ' cuts the tangents at P, P' in H, K and the corresponding directrix in N; prove that $\{HK; NS\}$ is harmonic.

99. Prove that conics having the same focus and directrix can be projected into concentric circles.

100. Two conics have a common focus and directrix; a chord PQ of one touches the other at H and meets the directrix at K; prove that $\{PQ; HK\}$ is harmonic.

101. Deduce a theorem for the parabola from: N is the foot of the perpendicular from a point P on an ellipse to a principal axis AA'; C is the centre; NQ is drawn parallel to AP and meets CP at Q; then AQ is parallel to the tangent at P.

102. If a system of conics pass through four given points, prove that the polars of a given point w.r.t. the system are concurrent.

What is the dual theorem?

By taking a special case of the dual theorem, obtain a property of the centres of the system of conics.

By projecting the dual theorem, obtain a property of confocal conics.

103. A conic passes through three fixed points and touches a fixed straight line; prove that the locus of the pole of the line joining two of the points is a conic touching each of the joins of the fixed points.

104. Prove that it is possible to project a conic and two conjugate lines into a parabola, its directrix and latus rectum.

105. Generalise by projection: the circumcircle of a triangle circumscribing a parabola passes through the focus of the parabola.

106. Prove that it is possible to project a conic, and a point off it, into a rectangular hyperbola and its focus.

107. If a conic and one of its foci is projected into a circle and its centre, prove that any angle at the focus is unaltered in magnitude by projection.

Use this method of projection to deduce angle properties of a conic from a circle. [*e.g.* Ex. 91, 96, 112, etc.]

108. Assuming that the locus of a point from which a parabola is viewed under a constant angle θ is a parabola having double contact with the first parabola, explain why the position of the points of contact is independent of the value of θ; and discuss the special case $\theta = 90°$.

109. Generalise by projection: the orthocentre of a triangle, circumscribing a parabola, lies on the directrix.

110. Generalise by projection: if a variable circle touches two fixed circles, centres A, B, then the locus of its centre is two conics, each having A, B as foci.

111. The foci of any curve are defined as the points from which two of the tangents to the curve are isotropic lines. Prove that the inverse of a focus of a curve is a focus of the inverse curve.

112. Two chords PQ, QR of a conic subtend equal angles at a focus; prove that PR meets the tangent at Q on the directrix.

113. A line drawn through the focus S of a hyperbola, parallel to an asymptote, meets the curve at Q; prove that $4SQ$ is equal to the latus rectum.

114. If in Ex. 113, SQ meets the other asymptote at R, prove that $4QR$ is equal to the length of the real axis.

115. The tangent at a point P on a hyperbola, focus S, meets an asymptote CT at T; SP meets CT at Q; prove that $QT=QS$.

THEOREM 83.

A system of conics, touching four straight lines, can be projected into a system of confocal conics.

It is only necessary to project one pair of opposite vertices of the circumscribing quadrilateral into the circular points at infinity; for, by definition, the projected system is then confocal.

Q.E.D.

Corollary.

Any two conics can be projected into two confocal conics.

[Any two conics have four common tangents.]

THEOREM 84.

A system of conics, having a common focus S and a common corresponding directrix L, can be projected into a system of concentric circles, having the projection of S as centre.

With the usual notation, let $S\omega$, $S\omega'$ cut L at P, Q.

Then since the conics have S as focus and L as directrix, by definition each touches $S\omega$, $S\omega'$ at P, Q.

If then P, Q are projected into the circular points at infinity each conic becomes a circle having the projection of S as centre.

Q.E.D.

NOTE ON METHOD.

For rider work, it is useful to bear in mind to what extent metrical properties can be transmitted by projection.

(1) It has been pointed out already that harmonic and cross-ratio properties of ranges or pencils are unaffected by projection.

(2) If A, B, C are three collinear points, the ratio $\frac{AB}{CB}$ can be transmitted to the projected figure, by writing it in the form $\{ABC\infty\}$, where ∞ denotes the ideal point on AB.

And in particular, if $AB=BC$, it must be restated in the form $\{ABC\infty\}$ is harmonic.

(3) The fact that an angle ABC is a right angle can be allowed for, by saying that the pencil $B\{AC;\ \omega\omega'\}$ is harmonic.

If the angle ABC is of constant magnitude, this must be re-written as $B\{A\omega C\omega'\}=$ constant.

(4) The condition that a conic is a circle is satisfied by saying that it passes through ω, ω'.

(5) The condition that a conic is a rectangular hyperbola requires that ω, ω' should be conjugate points w.r.t. it.

(6) The condition that a point S is a focus of a conic Σ is equivalent to the statement that $S\omega$, $S\omega'$ are tangents to Σ.

As has been already noted, it is frequently easier to see how to prove a theorem which is expressed in general, rather than special terms. Consequently it may be desirable sometimes first to generalise the given problem, and then project it afresh by a different method. The following example illustrates these remarks.

EXAMPLE.

Any circle through the centre C of a rectangular hyperbola cuts the curve at H, K; then one pair of common chords of the circle, and a conic through H, K, which circumscribes any triangle self-conjugate w.r.t. the hyperbola, are conjugate lines w.r.t. the hyperbola.

Now $C\omega\omega'$ is a triangle inscribed in the given circle S_1, and it is also self-conjugate w.r.t. the hyperbola Σ.

The given theorem is therefore a special case, by projection, of the following:

Each of two conics S_1, S_2 circumscribes a triangle self-conjugate w.r.t. a third conic Σ, and two of their common points H, K lie

on Σ; then one pair of their common chords are conjugate lines w.r.t. Σ.

Project this generalised form, so that H, K become the circular points at infinity.

We then obtain:

Each of two circles S_1, S_2 circumscribes a triangle self-conjugate w.r.t. a third circle Σ, then their radical axis is conjugate to the line at infinity w.r.t. Σ, or in other words, their radical axis passes through the centre of Σ.

This last theorem is easily proved.

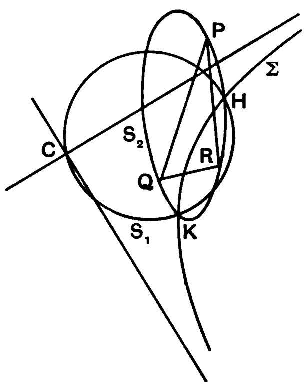

FIG. 65.

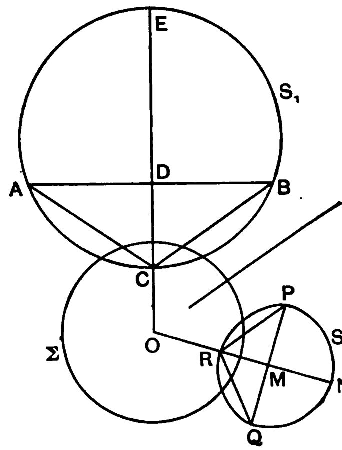

FIG. 66.

Let PQR, ABC be the two triangles, and O the centre, and r the radius of Σ.

Then O is the orthocentre of PQR and ABC.

Let OC meet AB at D and S_1 again at E.

Then by elementary theorems [Part I, Th. 23, 71.]

$$OD = DE \text{ and } OC \,.\, OD = r^2;$$

$$\therefore\ OC \,.\, OE = 2OC \,.\, OD = 2r^2,$$

$\therefore$ the square of the tangent from O to $S_1 = 2r^2$,

and similarly with S_2.

$\therefore$ O lies on the radical axis of S_1 and S_2. Q.E.D.

This theorem is a special case of Theorem 221.

116. Four conics pass through the points A, B, C, D; prove that the cross-ratio of the pencil formed by the tangents at A to the conics is the same as that at B or C or D.

117. From a fixed point T, the tangents TA, TB, are drawn to a fixed conic; PQ is a variable chord, such that $T\{PQ; AB\}$ is constant; find the envelope of PQ.

118. Generalise by projection: P is a variable point on a given circle; C is a fixed point; CP is produced to Q so that $\frac{CP}{CQ}$ is constant; then the locus of Q is another circle.

119. A system of conics have double contact with each other at two fixed points A, B; prove that the poles of a given line PQ w.r.t. the system lie on another line PR such that $P\{AB; QR\}$ is harmonic.

120. Generalise by projection: the nine-point circle of a triangle touches each of the four circles which touch the sides of the triangle.

121. Generalise by projection: the locus of the foci of ellipses which touch the four sides of a given parallelogram is a rectangular hyperbola, circumscribing the parallelogram.

122. Three conics S_1, S_2, S_3 have a common chord AB; prove that the locus of a point P, which is such that its polars w.r.t. S_1, S_2, S_3 are concurrent, is the line AB and a conic Σ through A, B. Further, if Σ meets S_1 in P and the tangent at P to S_1 in p, prove that $P, p; A, B$ form a harmonic system of points on Σ.

123. Generalise by projection: the locus of the centre of a conic inscribed in a given quadrilateral is the straight line through the mid-points of the three diagonals.

124. The base of a triangle passes through a fixed point A; the extremities of the base move on fixed tangents to a conic, and the remaining sides touch the conic; AP, AQ are the tangents from A to the conic; prove that the locus of the third vertex of the triangle is a conic having double contact with the given conic at P, Q.

125. If two triangles ABC, PQR are such that the sides of one are the polars of the vertices of the other w.r.t. a conic, prove that the triangles are in perspective. [Project the conic into a circle having the projection of A as centre.]

126. Generalise by projection: if a variable circle touches a fixed conic at a fixed point and cuts it again at P, Q, then PQ is fixed in direction.

127. Generalise by projection: if a circle has double contact with a conic, its chord of contact is parallel to one of the axes of the conic.

128. Generalise by projection: a variable conic is inscribed in a given triangle; if one focus lies on a fixed line, then in general, the other focus lies on a fixed conic.

If, however, the fixed line passes through a vertex of the triangle, the other focus lies on another straight line through the same vertex.

129. Generalise by projection: if two conics have the same directrix, then their common points are concyclic.

Deduce, by projecting the generalised form, a property of coaxal circles.

130. Two conics cut at A, B, C, D; a variable line through A meets the conics at P, Q; prove that $B\{PCQD\}$ is constant.

131. A, B are two fixed points; C is a fixed point on a given conic; PQ is a variable chord such that $C\{AB;\ PQ\}$ is harmonic; prove that PQ passes through a fixed point.

What theorem is obtained by taking A, B at ω, ω'?

132. A variable parabola touches the sides of a fixed triangle ABC; prove that each chord of contact passes through a fixed point.

What theorem is obtained by projecting B, C into ω, ω'?

133. AA′, BB′, CC′, DD′ are four concurrent chords of a conic; if P is any other point on the conic, prove that

$$\mathbf{P\{ABCD\}=P\{A'B'C'D'\}}.$$

134. The joins of the meet H of two common tangents of two conics to any four points A, B, C, D on one of the conics cut the other at A_1A_2, B_1B_2, C_1C_2, D_1D_2; P, P_1 are any two points on the conics, prove that, with an appropriate selection of points,

$$P\{ABCD\}=P_1\{A_1B_1C_1D_1\}=P_1\{A_2B_2C_2D_2\}.$$

135. Generalise by projection: a conic has double contact with a circle S_1 at A, B, and cuts a circle, concentric with S_1, at P, Q, R, S; then AB is parallel to PQ and RS, and is halfway between them.

136. p, q are two fixed lines, touching a given conic Σ; a variable conic S touches p, q and two other fixed lines; prove that the other two common tangents of S, Σ meet on a fixed line. [Project the dual theorem.]

137. S_1, S_2, S_3 are three conics inscribed in the same quadrilateral; P is a point of intersection of S_1 and S_2; prove that the tangents at P to S_1, S_2 are harmonic conjugates w.r.t. the tangents from P to S_3. [Project the dual theorem.]

138. A straight line cuts one conic in A, B and another conic in C, D. The tangents at A, B meet the tangents at C, D in F, G and H, K. If the conics cut each other at P, Q, R, S, prove that the eight points $PQRSFGHK$ lie on a conic.

139. Generalise by projection: O is the centre of a system of concentric circles; then the mid points of their chords of intersection with a given conic S lie on a rectangular hyperbola through O and the centre of S.

140. Generalise by projection: if, from two variable points on the directrix, which subtend a constant angle at the focus, tangents are

drawn to the conic, then they intersect on one of two conics having the same focus and directrix.

141. Three conics α, β, γ circumscribe a triangle DEF; β, γ; γ, α; α, β meet again at A, B, C; a fourth conic through E, F cuts α in G, H; β in K, L; γ in M, N; if the conics $ADEFG$, $BDEFK$, $CDEFM$ have four common points, prove that the conics $ADEFH$, $BDEFL$, $CDEFN$ also have four common points.

142. Generalise by projection: the director circles of all conics touching four fixed lines are coaxal. [The director circle of a conic is the locus of points, from which the tangents to a conic are at right angles; see Theorem 76.]

143. Generalise by projection: if a variable conic has double contact with each of two circles, centres A, B; then the locus of its centre is the line AB and the circle on AB as diameter.

144. A triangle is circumscribed to a given conic; two of its vertices move on fixed lines which meet at T; TP, TQ are the tangents from T to the conic; prove that the locus of the third vertex is a conic having double contact with the given conic at P, Q.

145. Prove that the asymptotes of all conics which touch two given straight lines at given points envelope a parabola.

146. H, K are fixed points on the tangent at a point F on a given conic; TP, TQ are a pair of variable tangents, such that $T\{PQ; HK\}$ is harmonic; find the locus of T.

147. $ABCP$ are four points on a hyperbola; two lines through P, parallel to the asymptotes, meet BC, CA, AB at L, M, N and L', M', N'; prove that $\dfrac{LM}{MN}=\dfrac{L'M'}{M'N'}$.

[Project A, C into ω, ω', and prove $B\{LMN\infty\}=B\{L'M'N'\infty'\}$.]

148. The chords of contact of two common tangents to two conics meet at H; prove that H is one of the diagonal points of the quadrangle formed by the points of intersection of the conics.

149. Generalise by projection: the envelope of the polars of a given point w.r.t. a system of confocal conics is a parabola touching the axes of the system.

150. A system of conics is inscribed in the triangle ABC, so as to touch BC at a fixed point D. Prove that the tangents at the points, in which any fixed line through D cuts the conics again, envelope a conic. [Project the dual theorem.]

151. A, B are fixed points on two of the sides of a given quadrilateral, the other tangents from A, B to any conic inscribed in the quadrilateral meet at P; prove that the locus of P is a straight line. [Project the dual theorem.]

152. TP, TQ are the tangents to two conics at one of their points of intersection; A, A' are any pair of opposite vertices of the quadrilateral formed by their common tangents; prove that $T\{PQ; AA'\}$ is harmonic. [Project the dual theorem.]

153. [**Hesse's Theorem.**] **If two pairs of opposite vertices of a complete quadrilateral are conjugate points w.r.t. a conic, prove that the third pair are also conjugate points.** [Part I., Th. 86 (3) and p. 161, Ex. 12.]

154. The asymptotes of two hyperbolas are parallel; prove that their common chord is parallel to one of the diagonals of the parallelogram formed by the asymptotes.

155. Generalise by projection: AB is a chord of a given circle, fixed in direction; a point P divides AB in a given ratio; then the locus of P is a conic having double contact with the circle.

156. Generalise by projection: a variable circle cuts a fixed circle orthogonally and touches a fixed straight line; then the locus of its centre is a parabola, and the envelope of the radical axis of the two circles is a conic.

157. Generalise by projection: a system of circles pass through two fixed points A, B; a fixed line AC through A meets any one of the circles at P; then the tangent at P envelopes a parabola whose focus is B.

Write down the dual of the generalised theorem, and deduce from it a theorem for a system of parabolas.

158. Generalise by projection: PQR is an equilateral triangle inscribed in a circle; three parallel lines through its vertices cut the circle again at P', Q', R'; then $P'Q'R'$ is an equilateral triangle, and the six sides of the two triangles touch a concentric circle.

159. A variable parabola touches the sides of a given triangle ABC at P, Q, R; AP, BQ, CR concur at O; prove that the locus of O is a conic circumscribing the triangle ABC. [Project B, C into ω, ω', note that $R\{AO; QP\} = -1$, and assume that the foot of the perpendicular from the focus of a parabola to any tangent lies on the tangent at the vertex.]

160. PQ is a variable chord, of fixed direction, of a parabola; find the locus of a point dividing PQ in a constant ratio.

161. Two conics have four-point contact at A; any line through A meets the conics at P, Q; prove that the locus of the meet of the tangents at P, Q is a straight line.

CHAPTER VI.

CELEBRATED PROPERTIES OF THE CONIC.

THE initial theorem of this chapter was first enunciated, in complete generality, by Pascal (1623-1662), at the age of sixteen, under the name of the Mystic Hexagram. Having established it first as a property of the circle, he then extended it, by projection, to the conic. The special case, however, where the conic degenerates to two straight lines is contained in the Porisms of Euclid unproved, and was formally proved by Pappus in his great treatise, six centuries later. But the credit of Pascal's discovery must be ascribed partly to Desargues, in whose work, as we shall see (page 318), the theorem is really implicitly contained, and upon whose ideas the fundamental principles of modern descriptive geometry are based. The dual theorem was discovered by Brianchon (1806), by an application of polar properties. It is interesting, as being the earliest professedly dual property, and consequently the forerunner of the important extensions made by Gergonne, Poncelet and Plücker. The property known as Pappus' theorem, or the "Locus ad quatuor lineas," has an interesting historical record. It is mentioned by Pappus as well-known: but no proof was given until Descartes solved it in 1659 by means of his discovery of the application of analysis to geometry. A purely geometrical solution was afterwards obtained by Newton, and was published in the *Principia*.

Evidence of the value of the ideas of Desargues and Pascal is readily obtained from the two great works of Carnot, the *Geometry of Position* (1803), and the *Theory of Transversals* (1806). Carnot took a leading political part in the revolutionary changes in France at the end of the eighteenth century, and after his banishment in 1796 devoted his time to mechanics and geometry. The

geometrical significance of negative number occupies a prominent place in his writings.

The remaining theorem of this chapter is associated with the name of Sir Isaac Newton* (1642-1727). He may be justly considered the greatest mathematical genius the world has ever known: and yet his researches were almost entirely completed before he reached the age of thirty, after which time he gave himself up to politics and theology. Owing to his dread of controversy, which publication in those days involved, it was with the greatest difficulty that he could be persuaded by his friends to announce his discoveries. The *Principia* itself was not communicated to the Royal Society until twenty years after it was written. Either this, or his work

THEOREM 85. [PASCAL'S MYSTIC HEXAGRAM.]

If a hexagon is inscribed in a conic, then the meets of the three pairs of opposite sides are collinear.

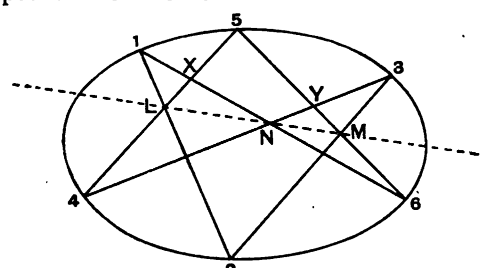

FIG. 67.

1, 2, 3, 4, 5, 6 are the vertices of the hexagon.

12, 45; 23, 56; 34, 61; are the pairs of opposite sides.

Let L, M, N be their meets; let X, Y be the meets of 16, 54; 34, 56.

†Now $1\{2456\} = 3\{2456\}$, by Chasles' theorem.

But $1\{2456\} = \{L45X\}$, base 45,

and $3\{2456\} = \{MY56\}$, base 56;

$\therefore \{L45X\} = \{MY56\}$.

But 5 is a common point of the two ranges;

$\therefore LM$, $4Y$, $X6$ are concurrent.

Now N is the meet of $4Y$, $X6$.

$\therefore L$, M, N are collinear. Q.E.D.

* It has been pointed out however by the Rev. J. J. Milne that this theorem was employed by Pappus, and in essence is due to Apollonius.

† In order to remember this proof, the reader should note that the vertices of the two pencils employed are two vertices of the hexagon, separated by *one* vertex.

on Fluxions, or his optical discoveries, would alone place him in the front rank. As a geometer, judged by the nature of his methods, he is to be classed among the last of the ancients rather than with the first of the modern school. The modesty of his nature, unaffected by the European fame of his achievements, is illustrated by the following extracts from his correspondence: "I seem to have been only like a boy playing on the sea shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me;" and "if I have seen farther than other men, it is only because I have stood on the shoulders of giants."

THEOREM 86. [BRIANCHON'S THEOREM.]

If a hexagon is circumscribed about a conic, then the joins of the three pairs of opposite vertices are concurrent.

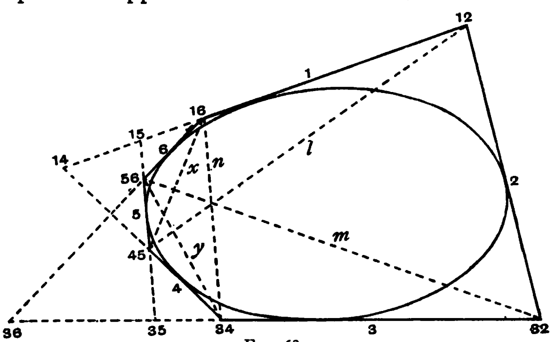

FIG. 68.

1, 2, 3, 4, 5, 6 are the sides of the hexagon.

12, 45; 23, 56; 34, 61; are the pairs of opposite vertices.

Let l, m, n be their joins; let x, y be the joins of 16, 54; 34, 56.

*Now $\quad 1\{2456\} = 3\{2456\}$, by Chasles' theorem.

But $\quad 1\{2456\} = \{l45x\}$, vertex 45,

and $\quad 3\{2456\} = \{my56\}$, vertex 56;

$\therefore \quad \{l45x\} = \{my56\}$.

But 5 is a common ray of the two pencils;

$\therefore$ lm, $4y$, $x6$ are collinear.

Now n is the join of $4y$, $x6$;

$\therefore$ l, m, n are concurrent. Q.E.D.

* In order to remember this proof, the reader should note that the bases of the two ranges employed are two sides of the hexagon, separated by *one* side.

PASCAL'S THEOREM (SECOND PROOF).

With the notation of Fig. 67, project the line LM to infinity and the conic into a circle.

Then in the new figure, 12, 45 and 23, 56 are parallel;

$$\therefore\ \widehat{143} = 180^\circ - \widehat{123}$$
$$= 180^\circ - \widehat{456} \text{ by parallels}$$
$$= \widehat{416};$$
$$\therefore\ 16 \text{ is parallel to } 43;$$

$\therefore$ the meets of 12, 45; 23, 56; 34, 61 lie on the line at infinity;

$\therefore$ in the original figure, L, M, N are collinear. Q.E.D.

Brianchon's Theorem may be also proved, very simply, by projection (see Ex. 3). In order to show the exact connection between the two theorems, we shall next derive Brianchon's property from Pascal's, by the use of poles and polars, which was the way in which it was actually discovered.

BRIANCHON'S THEOREM (SECOND PROOF).

Let $ABCDEF$ be the circumscribing hexagon; then the polars a, b, c, d, e, f of its vertices form an inscribed hexagon. Since a, d are the polars of A, D; the point ad is the pole of AD. Now by Pascal's theorem, the points ad, be, cf are collinear;

$\therefore$ their polars AD, BE, CF are concurrent. Q.E.D.

PASCAL'S THEOREM (THIRD PROOF).

The following method is due to Dr. Salmon.

Let a, b, c, d, e, f be the vertices of the inscribed hexagon.

Denote the equation of the join of a, b by $ab = 0$, etc.

Since the given conic circumscribes the quadrangles $abcd$, $afed$, its equation can be written in either of the forms,

$$ab \, . \, cd - \lambda \, . \, bc \, . \, ad = 0 \text{ or } de \, . \, fa - \mu ad \, . \, fe = 0,$$

so that $$ab \, . \, cd - \lambda \, . \, bc \, . \, ad \equiv \nu(de \, . \, fa - \mu ad \, . \, fe)$$

where λ, μ, ν are constants;

$$\therefore\ ab \, . \, cd - \nu de \, . \, fa \equiv ad(\lambda \, . \, bc - \mu\nu \, . \, fe).$$

Now the right-hand side has linear factors;

$$\therefore\ ab \, . \, cd - \nu de \, . \, fa = 0$$

must represent two straight lines, which pass through the points ab, de; ab, af or a; cd, de or d; cd, fa;

$\therefore$ the join of ab, de; cd, fa is the same as the line $\lambda bc - \mu\nu \, . \, fe = 0$;

$\therefore$ the points ab, de; cd, fa; bc, fe are collinear. Q.E.D.

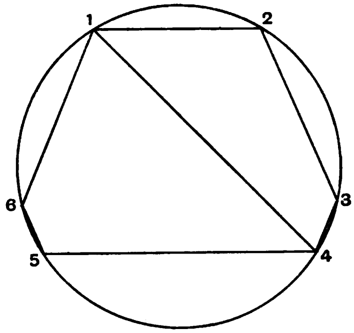

FIG. 69.

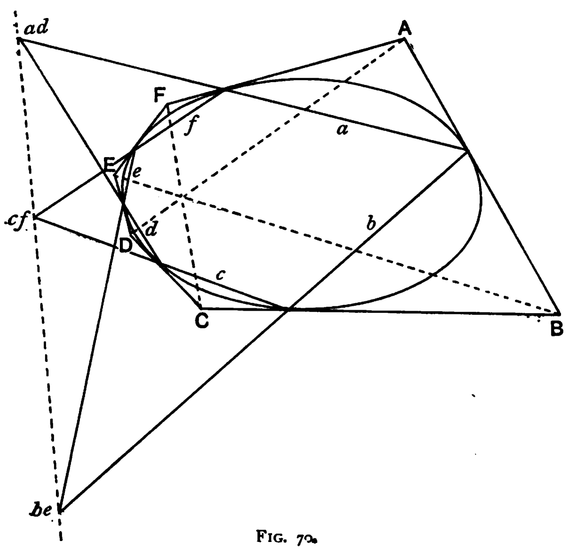

FIG. 70.

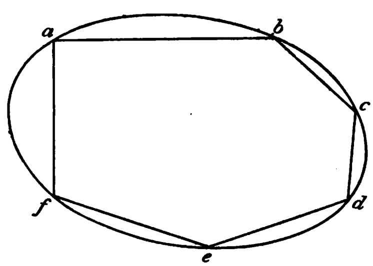

FIG. 71.

1. [STEINER'S THEOREM.] Prove that the Pascal lines of the hexagons *abcdef*, *adcfeb*, *afcbed* are concurrent. [The odd vertices are kept fixed, and the even vertices are permuted in cyclic order. Identify the two forms of the equation of the conic, given above, with

$$be\,.\,cf-\rho bc\,.\,ef=0.$$

The Pascal line of *abcdef* is $\lambda bc-\mu v\,.\,fe=0$; and similarly for the others.]

THEOREM 87.

(1) If the meets of the three pairs of opposite sides of a hexagon are collinear, then its six vertices lie on a conic.

(2) If the joins of the three pairs of opposite vertices of a hexagon are concurrent, then its six sides touch a conic.

The proof is left to the reader.

[It may be proved, merely by reversing the argument in the first method, or by a *reductio ad absurdum* method, assuming that one and only one conic can be drawn to pass through five given points, or to touch five given lines.]

2. Prove Theorem 87.

3. Prove Brianchon's theorem, by projecting the conic into a circle, having the projection of the meet of two of the diagonals as centre.

4. Prove that there are 60 Pascal lines associated with any six points on a conic.

5. Deduce a theorem from Pascal's property, by making two consecutive vertices of the hexagon coincide.

6. Deduce a property of a triangle inscribed in a conic from Pascal's theorem, by making consecutive vertices coincide, in pairs.

7. Deduce from Brianchon a property of a triangle circumscribing a conic.

8. If $ABCDEF$ is a hexagon inscribed in a conic, prove that the meets of AC, DF; DE, BC; AE, BF are collinear.

9. Prove that the 60 Pascal lines associated with six points on a conic form sets of four concurrent lines.

What is the dual theorem?

10. If the Pascal line of the hexagon $ABCDEF$ meets the conic at P, Q, prove that $A\{AECP\}=A\{DBFP\}$.

11. Given six collinear points A, E, C, D, B, F, construct a point P on the line, such that $\{AECP\}=\{DBFP\}$. Prove that there are two such positions of P, real, coincident or imaginary. [Use Ex. 10.]

12. Four points P, Q, R, S are taken on a conic; PR meets QS at L; M is any point; PM, QM meet the conic at T, U; prove that ST, RU, LM are concurrent.

13. Deduce from Brianchon's theorem, a property of a pentagon circumscribing a parabola.

14. *ABCDEF* is a hexagon circumscribing a conic; deduce from Brianchon's theorem, a property by making *AB*, *BC* and *DE*, *EF* coincide.

15. What does Brianchon's theorem become, if two consecutive sides of a hexagon, circumscribing a conic, coincide with the line at infinity?

16. *ABCD* is a fixed quadrilateral circumscribing a given conic; two variable tangents *PQ*, *RS* meet *BC*, *DC*, *BA*, *DA* at *P*, *Q*, *R*, *S* respectively; prove that *PS* meets *QR* on a fixed line.

17. A variable conic passes through four fixed points *A*, *B*, *C*, *D* and cuts two fixed lines *AP*, *AQ* at *P*, *Q*; if *CP* meets *BQ* at *N*, prove that the locus of *N* is a straight line.

DEDUCTIONS FROM THE PASCAL-BRIANCHON PROPERTIES.

THEOREM 88.

(1) If *A*, *C*, *E*; *B*, *D*, *F* are two sets of three collinear points; then the meets of *AD*, *BC*; *CF*, *ED*; *AF*, *EB* are collinear.

(2) If *a*, *c*, *e*; *b*, *d*, *f* are two sets of three concurrent lines; then the joins of *ad*, *bc*; *cf*, *ed*; *af*, *eb* are concurrent.

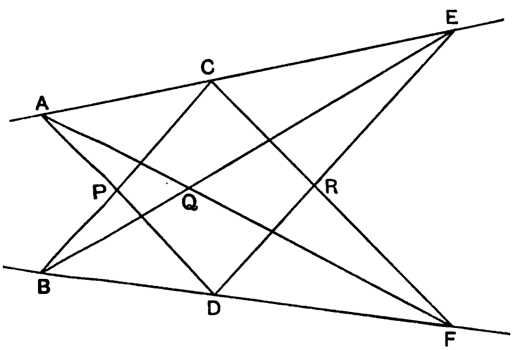

FIG. 72.

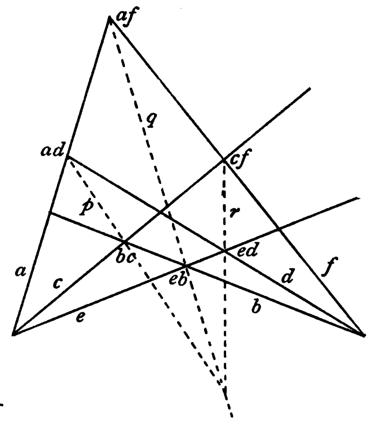

FIG. 73.

(1) The hexagon *ADEBCF* is inscribed in the conic formed by the lines *ACE*, *BDF*; and therefore the meets of the opposite sides are collinear. (Q.E.D.)

(2) This is simply the dual of (1).

It may be deduced independently, from Brianchon, by regarding the **finite** line joining the vertices of the pencils *a*, *c*, *e*; *b*, *d*, *f*, as a conic—a flattened ellipse.

Another proof is given, in Theorems 122, 123.

THEOREM 89.

If a quadrangle is inscribed in a conic, the tangents at its vertices meet in pairs on the sides of the diagonal point triangle.

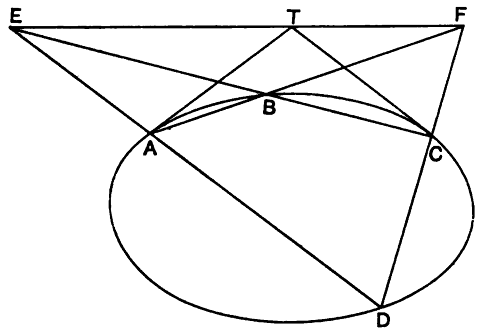

FIG. 74.

The proof is left to the reader.
[Consider the hexagon $AABCCD$.]

THEOREM 90.

If a pentagon *abcef* circumscribes a conic, and if *ce* touches the conic at *d*, then *ad*, *be*, *cf* are concurrent.

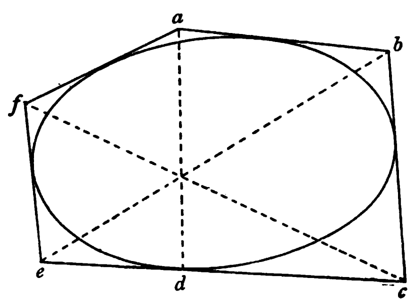

FIG. 75.

The proof is left to the reader.
[Consider the hexagon *abcdef*.]

18. Prove Theorem 89.

19. Prove Theorem 90.

20. A, B, C, A', B', C' are six points in a plane, such that the pairs of lines AB', $A'B$; BC', $B'C$; CA', $C'A$ are parallel. If A', B', C', are collinear, prove that A, B, C are also collinear.

21. T is the pole of a chord PQ of a conic; PH, QK are chords parallel to TQ, TP; prove that PQ is parallel to HK.

22. If a triangle is inscribed in a conic, prove that the tangents at the vertices meet the opposite sides in three collinear points.

23. P, Q, R are three points on a hyperbola; QR meets the line through P parallel to one asymptote at S; PQ meets the line through R parallel to the other asymptote at T; prove that ST is parallel to the tangent at Q.

24. P, Q, R, S are four points on a parabola; the diameters through Q, R meet PR, QS at H, K; prove that HK is parallel to PS.

25. The sides O_1O_2, O_2O_3, O_3O_1 of the triangle $O_1O_2O_3$ pass respectively through the vertices U_3, U_1, U_2 of the triangle $U_1U_2U_3$; A_1 is any point on U_2U_3; O_3A_1, O_2A_1 meet U_1U_3, U_1U_2 at A_2, A_3; prove that O_1, A_2, A_3 are collinear.

26. P, Q, R, T are four points on a conic; QR, PT meet the tangents at T, R in C, D; prove that CD, PQ, RT are concurrent. By taking RT as the line at infinity, deduce a theorem for a hyperbola.

27. B is a fixed point on a parabola; diameters are drawn through the extremities P, Q of a variable chord to meet BQ, BP at L, M; prove that LM is fixed in direction.

28. The corresponding sides of two triangles inscribed in a conic are parallel; prove that the joins of corresponding vertices are diameters.

29. B is a fixed point on a conic; BP, BQ are two variable chords; PH, QK are two chords parallel to BQ, BP; prove that HK is fixed in direction.

30. P is a variable point on a fixed line L; C, E are two fixed points and b, d two fixed lines; X, Y, Z are the meets of b, d; b, CE; d, CP; prove that the conic through E, P, X, Y, Z cuts L at a fixed point.

31. ABC is a triangle; three concurrent lines AO, BO, CO meet BC, CA, AB at A', B', C'; any line through A meets $A'B'$, $A'C'$ at B'', C''; prove that BB'', CC'', $B'C'$ are concurrent.

32. Any straight line divides a quadrilateral into two other quadrilaterals; prove that the points of intersection of the internal diagonals of the three quadrilaterals are collinear.

33. A parabola is inscribed in a triangle ABC, touching BC at D; if the parallelogram $BACN$ is completed, prove that DN is a diameter.

34. A conic is inscribed in a triangle, touching one side at its midpoint; prove that the centre of the conic lies on a median.

35. A conic inscribed in the triangle ABC touches AB, AC at H, K; any other tangent to the conic cuts BC, AC at P, Q; prove that BK, QH, AP are concurrent.

36. The sides AB, BC, CD, DA of a quadrilateral touch a conic at P, Q, R, S; CP, CS meet AQ, AR at X, Y; prove that XY passes through B and D.

37. T is the pole of a chord PQ of a parabola; R is any other point on the curve; the diameters through P, Q meet QR, PR at H, K; prove that T is the mid-point of HK.

38. A, B, C are three points on a hyperbola; BC meets one asymptote at D; a straight line AE is drawn parallel to this asymptote to cut a line through D parallel to AB at E. Prove that CE is parallel to the other asymptote.

39. A conic touches the sides MN, NL, LM of a triangle at P, Q, R; LP, MQ, NR meet at O; parallels through O to MN, NL, LM meet QR, RP, PQ at X, Y, Z; prove that X, Y, Z are collinear.

40. PQ is a chord of a parabola; through a point T on the tangent at P a line is drawn parallel to PQ meeting the diameter through P at H; HQ meets the curve again at R; prove that TR is a diameter.

41. ABC, PQR are two triangles inscribed in a conic; AB, BC, CA meet QR, RP, PQ at P_1, P_2, P_3; Q_1, Q_2, Q_3; R_1, R_2, R_3; prove that P_1Q_2 is a Pascal line for the six points A, B, C, P, Q, R.

42. Deduce from Pascal's theorem that if a rectangular hyperbola circumscribes a triangle ABC, it passes through the orthocentre. [Let α, β be the points at infinity on the curve, and let the perpendicular from A to BC cut the curve at H; consider the hexagon $BAH\alpha\beta C$.]

43. ABC, DEF are two triangles inscribed in a parabola; if AD, BE, CF are respectively parallel to BC, CA, AB, prove that the join of the centroids of ABC, DEF is a diameter of the parabola.

44. A, B, C, D, E, F are six points such that the meets of AF, EC; BF, CD; AD, BE are collinear; prove that the meets of AE, CF; DF, BE; BC, AD are also collinear.

45. From any point on the circumcircle of a triangle ABC, perpendiculars are drawn to BC, CA, AB and meet the circle again at D, E, F; prove that a conic can be drawn to touch AB, BC, CD, DE, EF, FA.

46. If two triangles are in perspective, prove that the six meets of the sides of one with the non-corresponding sides of the other lie on a conic.

47. A hexagon circumscribes a conic S_1 and is inscribed in a conic S_2; prove that its Pascal line w.r.t. S_2 and its Brianchon point w.r.t. S_1 form a side and vertex of the common self-conjugate triangle of S_1 and S_2.

48. Given five tangents to a conic, construct their points of contact.

49. What does Brianchon's theorem become, if two of the sides of the circumscribing hexagon are asymptotes of the conic?

50. If a conic touches AB, BC, CD, DA at P, Q, R, S, prove that AC, BD, PR, QS are concurrent.

51. The tangents at two points P, Q on a hyperbola cut one asymptote at H, K and the other at H', K'; prove that HK', $H'K$, PQ are parallel.

52. T is the pole of a chord PQ of a hyperbola; TP, TQ meet one asymptote at H, K; KP cuts QH at N; prove that TN is parallel to HK.

53. A tangent at a point P on a hyperbola, centre C, cuts the asymptotes at H, K; prove that CP passes through the meet of the lines drawn through H, K parallel to the asymptotes, and deduce that $HP=PK$.

54. H, K are points on the asymptotes CH, CK of a hyperbola; the tangents from H, K to the curve meet at T; the tangent at any other point P on the curve cuts TH, TK at L, M; prove that HM, KL, CP are concurrent.

55. Prove that a conic can be drawn to touch the sides of a triangle at the points where the altitudes meet the opposite sides.

56. If two triangles are in perspective, prove that the six straight lines joining the vertices of one to the non-corresponding vertices of the other envelope a conic.

57. The sides of a parallelogram $ABCD$ are parallel to the asymptotes of a conic; prove that BA, BC and DA, DC meet the conic at the extremities of two chords which meet on AC.

Constructions.

A number of important constructions can be effected by means of Pascal's and Brianchon's theorems. The general method is illustrated by the following examples. The reader should note that the underlying principle is to build up one or other of the fundamental figures, Fig. 67 or Fig. 68.

EXAMPLE I.

Given five points on a conic, to construct any number of other points on the conic.

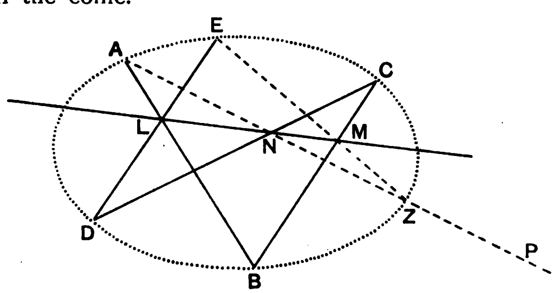

FIG. 76.

Let $ABCDE$ be the given points.

Draw through A, any line AP; it is required to find the point at which AP cuts the conic through A, B, C, D, E.

Let AB, DE meet at L and CD, AP at N.

Join LN and produce it to cut BC at M.

Join EM and produce it to cut AP at Z.

Then the meets of the opposite sides of the hexagon $ABCDEZ$ are collinear.

$\therefore$ the points A, B, C, D, E, Z lie on a conic. [Th. 87.]

$\therefore$ Z is the required point. Q.E.F.

EXAMPLE II.

Given five points on a conic, to construct the tangent to the conic at any one of them.

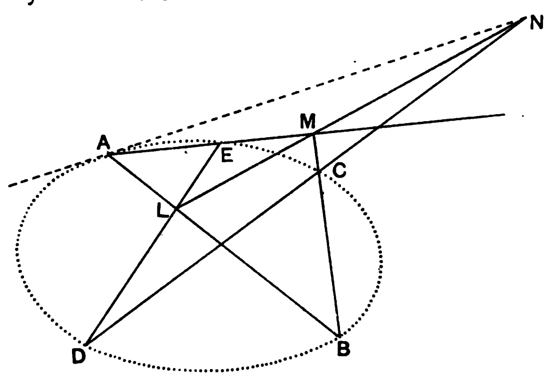

FIG. 77.

Let A, B, C, D, E be the given points.

[We shall construct the Pascal line of the hexagon $AABCDE$.]

Let AB, DE meet at L, and BC, EA at M.

Join LM and produce it to cut DC at N. Join NA.

Then NA is the tangent at A.

The proof is left to the reader. Q.E.F.

EXAMPLE III.

Given five tangents to a conic, to construct their points of contact.

This is immediately effected by using Theorem 90.

EXAMPLE IV.

Given three points on a hyperbola and one asymptote, to construct any number of points on the curve.

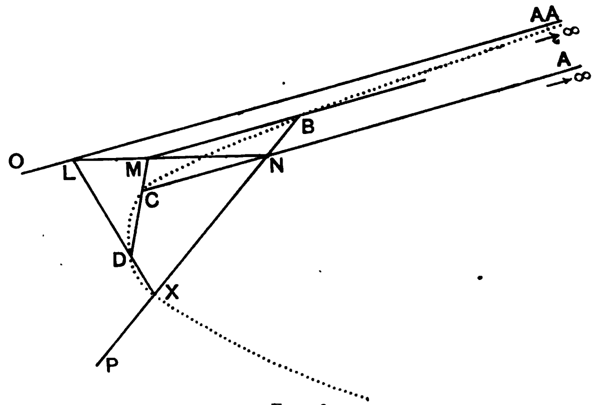

FIG. 78.

Let B, C, D be the given points, and A the point at infinity on the given asymptote OA. Draw any line BP through B.

It is required to find the point X in which BP cuts the curve.

[We shall construct the Pascal line of the hexagon $AABXDC$.]

The meet M of AB, DC is the meet of DC with a line through B parallel to OA.

The meet N of BP, CA is the meet of BP with a line through C parallel to OA.

Join MN and produce it to cut OA at L.

Join LD and produce it to cut BP at X.

Then X is the required point.

The proof is left to the reader. Q.E.F.

58. Prove Example II.

59. Prove Example IV.

60. Given four points on a hyperbola, and the direction of one asymptote, construct another point on the curve.

61. Given three points on a hyperbola and the directions of both asymptotes, construct (1) another point on the curve, (2) one of the asymptotes.

62. Given four points on a hyperbola and the direction of one asymptote, find the direction of the other.

63. Given four points on a hyperbola and the direction of one asymptote, construct that asymptote.

64. Given five points on a conic, construct its centre.

65. Given three points on a parabola and a diameter, construct the tangent at one of the given points.

66. Given three points on a hyperbola and one asymptote, construct the tangent at one of the given points.

67. Given five points on a conic, construct the polar of another given point.

68. Given four points on a conic and the tangent at one of them, construct the pole of a given line.

69. Given four tangents to a parabola, construct their points of contact.

70. Given three points on a conic and the tangents at two of them, construct the tangent at the third point.

71. Given two points on a parabola and the tangent at one of them and the direction of the axis, construct the tangent at the other point.

72. Given four tangents to a conic and the point of contact of one of them, construct the point of contact of one of the others.

73. Given one asymptote of a hyperbola, two tangents, and the point of contact of one of them, construct the point of contact of the other.

74. Given the asymptotes of a conic and a point on the conic, construct the tangent at that point.

75. A parabola is inscribed in a given triangle and its axis is given in direction, find its point of contact with one of the sides of the triangle.

THEOREM 91. [CARNOT'S THEOREM.]

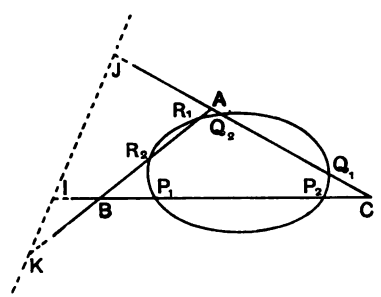

FIG. 79.

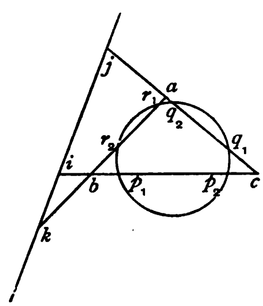

FIG. 80.

If the sides BC, CA, AB of a triangle cut a conic at P_1, P_2; Q_1, Q_2; R_1, R_2 respectively; then

$$\frac{AR_1 \cdot AR_2}{AQ_1 \cdot AQ_2} \cdot \frac{BP_1 \cdot BP_2}{BR_1 \cdot BR_2} \cdot \frac{CQ_1 \cdot CQ_2}{CP_1 \cdot CP_2} = 1.$$

Denote by I, J, K, the points at infinity on BC, CA, AB.

Project the conic into a circle, and use small letters for corresponding points in the new figure.

I, J, K project into three collinear points i, j, k.

Now $$\frac{BP_1}{CP_1} = \{BP_1CI\}, \text{ since } I \text{ is at infinity,}$$
$$= \{bp_1ci\} = \frac{bp_1}{cp_1} \cdot \frac{ci}{bi}.$$

Similarly $$\frac{BP_2}{CP_2} = \frac{bp_2}{cp_2} \cdot \frac{ci}{bi};$$

$$\therefore \frac{BP_1 . BP_2}{CP_1 . CP_2} = \frac{bp_1 . bp_2}{cp_1 . cp_2} \cdot \frac{ci^2}{bi^2}.$$

Similarly $$\frac{CQ_1 . CQ_2}{AQ_1 . AQ_2} = \frac{cq_1 . cq_2}{aq_1 . aq_2} \cdot \frac{aj^2}{cj^2},$$

and $$\frac{AR_1 . AR_2}{BR_1 . BR_2} = \frac{ar_1 . ar_2}{br_1 . br_2} \cdot \frac{bk^2}{ak^2}.$$

But $$bp_1 . bp_2 = br_1 . br_2, \text{ etc.};$$

$$\therefore \text{ the given expression} = \frac{ci^2}{bi^2} \cdot \frac{aj^2}{cj^2} \cdot \frac{bk^2}{ak^2}$$
$$= 1, \text{ by Menelaus' theorem,}$$

since i, j, k are collinear. Q.E.D.

Corollary.

If P_1, P_2; Q_1, Q_2; R_1, R_2 are points on the sides BC, CA, AB of a triangle, such that $\dfrac{AR_1 . AR_2}{AQ_1 . AQ_2} \cdot \dfrac{BP_1 . BP_2}{BR_1 . BR_2} \cdot \dfrac{CQ_1 . CQ_2}{CP_1 . CP_2} = 1$, then these six points lie on a conic.

The proof is left to the reader.

[Use a *reductio ad absurdum* method.]

76. Prove Theorem 91, Corollary.

77. What does Carnot's theorem become, if the conic degenerates into a straight line and the line at infinity; or into two coincident straight lines?

78. Extend Carnot's theorem to apply to a quadrilateral.

79. Extend Carnot's theorem to apply to any polygon.

80. If a conic touches the sides BC, CA, AB of a triangle at P, Q, R, prove that $\dfrac{BP}{PC} \cdot \dfrac{CQ}{QA} \cdot \dfrac{AR}{RB} = 1$.

81. By applying Carnot's theorem to the triangle formed by the asymptotes and one tangent of a hyperbola, deduce a property of the hyperbola.

82. With the notation of Carnot's theorem, prove that

$$\sin BAP_1 . \sin BAP_2 . \sin CBQ_1 . \sin CBQ_2 . \sin ACR_1 . \sin ACR_2$$
$$= \sin CAP_1 . \sin CAP_2 . \sin ABQ_1 . \sin ABQ_2 . \sin BCR_1 . \sin BCR_2 .$$

83. The in-circle of an isosceles triangle ABC touches BC, CA, AB at D, E, F; prove that the mid-points of AE, AF, BF, BD, CD, CE lie on a conic.

84. If two conics are drawn to touch three straight lines, prove that the six points of contact lie on a conic.

85. O, O' are two points in the plane of the triangle ABC; prove that the lines joining the vertices to O, O' meet the opposite sides in six points, situated on a conic.

86. Deduce from Carnot's theorem that, if a chord PQ of a hyperbola meets the asymptotes at H, K, then $HP=KQ$.

87. The tangent at any point D of a hyperbola cuts an asymptote BE at E, and a line BH parallel to the other asymptote at H. If BH cuts the hyperbola at F, prove that $\dfrac{HD^2}{ED^2}=\dfrac{HF}{BF}$.

88. A conic cuts the sides BC, CA, AB of a triangle at A_1, A_2; B_1, B_2; C_1, C_2; if AA_1, BB_1, CC_1 are concurrent, prove that AA_2, BB_2, CC_2 are also concurrent.

89. Through any point O, lines are drawn parallel to the sides of a triangle ABC; prove that their six meets with the sides lie on a conic.

90. Three straight lines meet the sides BC, CA, AB of a triangle at A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3; if A_1, B_2, C_3 are collinear, prove that A_2, A_3, B_3, B_1, C_1, C_2 lie on a conic.

91. Tangents are drawn from the vertices A, B, C of a triangle to a circle, and meet the opposite sides at P_1, P_2; Q_1, Q_2; R_1, R_2. Prove that these six points lie on a conic.

Generalise this theorem; and enunciate the dual property.

[Let O be the centre, and r the radius of the circle; note that $\sin BAP_1 . \sin BAP_2 = \sin^2 BAO - \sin^2 P_1AO = \dfrac{OF^2-r^2}{OA^2}$, where OF is the perpendicular from O to AB; and use Ex. 82.]

92. By making R_1, Q_1, Q_2 in Fig. 79 coincide with A, deduce the following: a conic touches AC at A and passes through three given points P_1, R_2, P_2; its circle of curvature at A cuts AR_2 at ρ; P_1P_2 cuts AR_2, AC at B, C; prove that $A\rho = AR_2 . \dfrac{BP_1 . BP_2}{CP_1 . CP_2} . \dfrac{CA^2}{BA . BR_2}$. [The circle of curvature at a point O on a conic is defined as the circle which cuts the conic in three consecutive points at O.]

93. A conic touches AC at A; P_1P_2 is a chord of the conic, parallel to AC; the circle of curvature at A cuts a chord AR_2 of the conic at ρ; P_1P_2 cuts AR_2 at B; prove that $\frac{A\rho}{AR_2}=\frac{BP_1 \cdot BP_2}{BA \cdot BR_2}$.

94. Deduce a theorem from Ex. 93, by taking B at the centre of the conic.

THEOREM 92. [PAPPUS' THEOREM.]

A, B, C, D are four fixed points on a given conic; *P* is a variable point on the conic; *PH, PK, PX, PY* are the perpendiculars from *P* to *AB, CD, AD, BC*; then $\frac{PH \cdot PK}{PX \cdot PY}$ is constant.

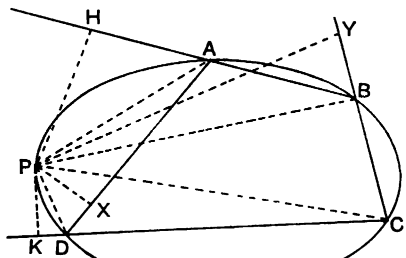

FIG. 81.

Now $\qquad PH \cdot AB = 2\triangle APB = PA \cdot PB \sin APB$;

similarly $\qquad PK \cdot CD = PC \cdot PD \sin CPD$;

$PY \cdot CB = PC \cdot PB \sin CPB$; $\quad PX \cdot AD = PA \cdot PD \sin APD$;

$$\therefore \frac{PH \cdot PK}{PX \cdot PY} \cdot \frac{AB \cdot CD}{AD \cdot CB} = \frac{\sin APB \cdot \sin CPD}{\sin APD \cdot \sin CPB} = P\{ABCD\};$$

$$\therefore \frac{PH \cdot PK}{PX \cdot PY} = \frac{AD \cdot CB}{AB \cdot CD} \cdot P\{ABCD\} = \text{constant.} \quad \text{[Th. 57.]}$$

Q.E.D.

Corollary.

A, B, C, D are four fixed points; from a variable point *P*, perpendiculars *PH, PK, PX, PY* are drawn to *AB, CD, AD, BC*; if $\frac{PH \cdot PK}{PX \cdot PY}$ is constant, then the locus of *P* is a conic through *A, B, C, D*.

[Use the method of Theorem 92.]

THEOREM 93.

AB, BC, CD, DA are four fixed tangents to a given conic; p is a variable tangent; *AH, BX, CK, DY* are the perpendiculars from *A, B, C, D* to p; then $\frac{AH.CK}{BX.DY}$ is constant.

The proof is left to the reader.

[Let p' be any other tangent; let p, p' meet AB in Q, Q' and CD in R, R'; let AH', BX', CK', DY' be the perpendiculars to p'; prove that $\frac{AH}{BX}=\frac{AQ}{QB}$, $\frac{CK}{DY}=\frac{CR}{DR}$, and note that

$$\{AQBQ'\}=\{DRCR'\}.]$$

Theorem 93 is due to Chasles.

95. Prove Theorem 92, Corollary.

96. Prove Theorem 93.

97. AH, BX, CK, DY are the perpendiculars from four fixed points A, B, C, D to a variable line p; $\frac{AH.CK}{BX.DY}$ is constant; prove that p envelopes a conic, touching AB, BC, CD, DA.

98. *PL, PM, PN* are the perpendiculars from a variable point *P* to the sides of a fixed triangle; if $\frac{PL^2}{PM.PN}$ is constant, find the locus of *P*.

99. *PA, PB, PC, PD, PE, PF* are the perpendiculars from a variable point *P* on a conic to consecutive sides of a given hexagon, inscribed in the conic; prove that $\frac{PA}{PB}\cdot\frac{PC}{PD}\cdot\frac{PE}{PF}$ is constant.

100. Extend Pappus' theorem to a polygon of $2n$ sides inscribed in a conic.

101. *P* is a variable point on a hyperbola; *PH, PK* are the perpendiculars to the asymptotes; prove that $PH.PK$ is constant.

If a parallelogram *MN* is formed by drawing lines through *P* parallel to the asymptotes to cut them at *M, N*, prove that *MN* is of constant area.

102. Deduce from Pappus' theorem a property of the parabola, by taking *B, C* as coincident points at infinity on the curve.

103. *AB* is a fixed chord of a hyperbola; parallels through *A, B* to the asymptotes meet at *C*; *PH, PK, PL* are the perpendiculars from a variable point *P* on the curve to *CA, CB, AB*; prove that $\frac{PH.PK}{PL}$ is constant.

104. With the notation of Fig. 81, if the conic is a circle, prove that PH . PK = PX . PY.

105. The tangents at the vertices of a triangle ABC inscribed in a conic form a triangle XYZ; P is a variable point on the curve, prove that the product of the lengths of the perpendiculars from P to BC, CA, AB is proportional to that of the perpendiculars from P to YZ, ZX, XY.

Generalise this theorem.

106. Extend Theorem 93 to a $2n$-sided polygon circumscribing a conic.

107. With the notation of Theorem 93, if the conic is a parabola, prove that AH . CK = BX . DY.

108. A is the pole of a chord BC of a conic; a, b, c are the perpendiculars from A, B, C to a variable tangent to the conic; prove that $\frac{a^2}{bc}$ is constant.

109. A variable line cuts two fixed lines OA, OB at P, Q; if the triangle OPQ is of constant area, find the locus of the mid-point of PQ.

110. The tangent at a fixed point B of a hyperbola meets the asymptotes, OC, OD at C, D; b, c, d, o are the lengths of the perpendiculars from B, C, D, O to a variable tangent; prove that $\frac{b \,.\, o}{c \,.\, d}$ is constant.

111. $ABCD$ is a fixed parallelogram inscribed in a conic; P is a variable point on the curve; parallels through P to the sides meet AB, CD in L, M and BC, AD in H, K; prove that $\frac{PL \,.\, PM}{PH \,.\, PK}$ is constant.

112. A conic touches the side BC of a triangle ABC at A'; t is a variable tangent; prove that the product of the distances of A and A' from t is proportional to the product of the distances of B and C from t.

113. A conic touches AB, AC at B, C; if ABC is the triangle of reference, prove that, in areal coordinates, the equation of the conic is $\eta\zeta = \lambda \,.\, \xi^2$. By taking AB, AC as the isotropic lines, deduce Theorem 82.

THEOREM 94. [NEWTON'S THEOREM.]

If POQ, ROS are two variable chords of a conic, fixed in direction, then the ratio $\dfrac{OP.OQ}{OR.OS}$ is constant.

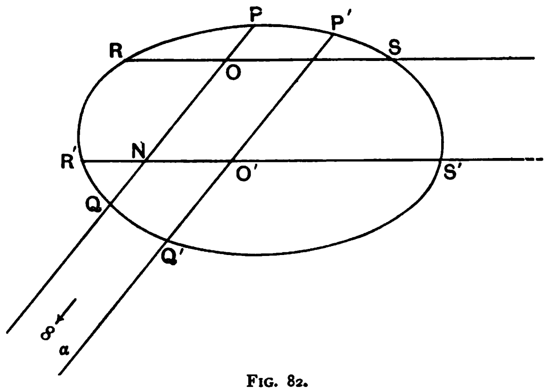

FIG. 82.

Take any other position O' of O.

Draw the chords $P'O'Q'$, $R'O'S'$ in the given directions.

Let $R'S'$ cut PQ at N; PQ, $P'Q'$ meet at a point at infinity α.

Since Q, Q', P, P' are finite points, $\dfrac{\alpha Q}{\alpha Q'} = 1 = \dfrac{\alpha P}{\alpha P'}$.

Apply Carnot to the triangle $\alpha NO'$.

Then $$\frac{NP.NQ}{NR'.NS'} \cdot \frac{\alpha Q'.\alpha P'}{\alpha Q.\alpha P} \cdot \frac{O'R'.O'S'}{O'P'.O'Q'} = 1;$$

$$\therefore \frac{O'P'.O'Q'}{O'R'.O'S'} = \frac{NP.NQ}{NR'.NS'}.$$

Similarly $$\frac{OP.OQ}{OR.OS} = \frac{NP.NQ}{NR'.NS'};$$

$$\therefore \frac{OP.OQ}{OR.OS} = \frac{O'P'.O'Q'}{O'R'.O'S'}.$$

Q.E.D.

The standard equations of the ellipse, hyperbola, and parabola can be deduced from Newton's theorem, by making use of the fact (see page 134) that the ellipse and hyperbola have one pair of perpendicular conjugate diameters; while the parabola has one diameter which bisects all chords perpendicular to it.

I. THE ELLIPSE AND HYPERBOLA.

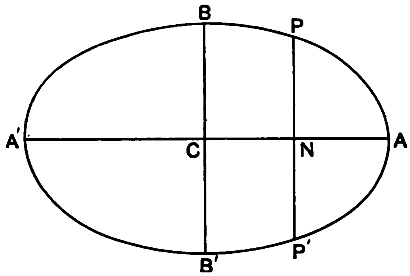

FIG. 83.

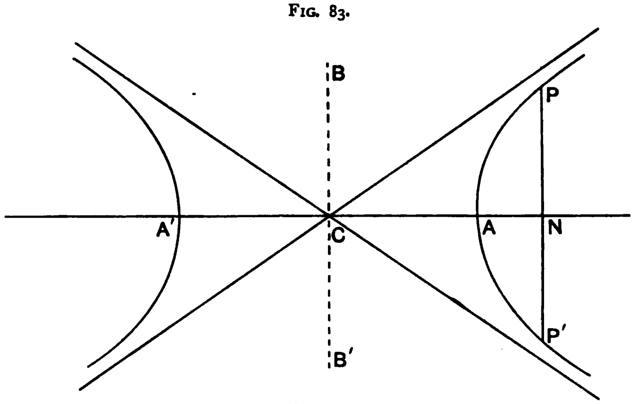

FIG. 84.

Let ACA', BCB' be the pair of perpendicular conjugate diameters. Draw the double ordinate PNP', of any point P on the curve, to AA'. Then, by Newton's theorem,

$$\frac{NP \cdot NP'}{NA \cdot NA'} = \frac{CB \cdot CB'}{CA \cdot CA'};$$

$$\therefore \frac{PN^2}{A'N \cdot NA} = \frac{CB^2}{CA^2}, \text{ since } PN = NP', \text{ etc.};$$

$$\therefore \frac{PN^2}{CA^2 - CN^2} = \frac{CB^2}{CA^2}, \text{ since } A'C = CA.$$

Let $CN = x$, $NP = y$, $CA = a$, $CB = b$;

Then
$$\frac{y^2}{a^2 - x^2} = \frac{b^2}{a^2};$$

$$\therefore a^2y^2 = a^2b^2 - b^2x^2;$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the standard form. Q.E.D.

From Chapter I., we know that this meets the line at infinity, where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$, so that, for the hyperbola, either a^2 *or* b^2 is negative; while for the ellipse, a^2 and b^2 are positive.

II. THE PARABOLA.

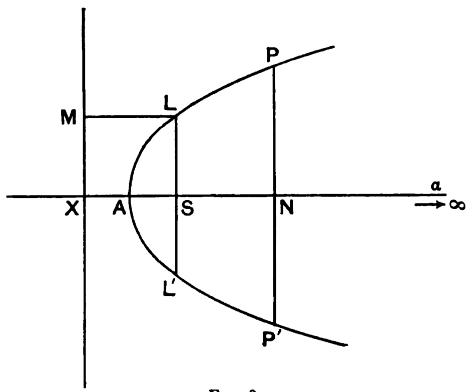

FIG. 85.

Let S be the focus, A the vertex, and LSL' the latus rectum of the parabola: and let α be the point at infinity on AS. PNP' is any double ordinate to the principal diameter AS. Draw LM perpendicular to the directrix.

By Newton's theorem,

$$\frac{NP \,.\, NP'}{NA \,.\, N\alpha}=\frac{SL \,.\, SL'}{SA \,.\, S\alpha}.$$

But $$\frac{S\alpha}{N\alpha}=1; \quad \therefore \quad \frac{PN^2}{AN}=\frac{SL^2}{AS}.$$

Now $$SL=LM=SX=2SA;$$

$$\therefore \quad \frac{PN^2}{AN}=\frac{4AS^2}{AS}=4AS;$$

$$\therefore \quad PN^2=4AS \,.\, AN.$$

Let $AN=x$, $NP=y$, $AS=a$.

Then $$y^2=4ax,$$

which is the standard form. Q.E.D.

114. What value of the constant is obtained in Theorem 94, by taking O at the centre of the conic?

Hence deduce Carnot's theorem from Newton's theorem.

115. PQ is a variable chord of a parabola, fixed in direction; the diameter bisecting PQ cuts the curve at V and PQ at N; prove that $\frac{PN^2}{VN}$ is constant.

116. PQ, HK are two parallel chords of a hyperbola, meeting an asymptote at Y, Z respectively; prove that $YP.YQ=ZH.ZK$.

117. T is the pole of a chord PQ of a conic, centre C; HK is another chord such that CH, CK are parallel to TP, TQ and drawn in the same sense; prove that HK is parallel to PQ.

118. Two diameters of a parabola meet the curve at H, K, and a chord PQ at H', K'; prove that $\frac{HH'}{KK'}=\frac{PH'.H'Q}{PK'.K'Q}$.

119. P is a point on an ellipse, centre C, axes ACA′, BCB′; the perpendicular from P to AA′ meets AA′ in N and the circle on AA′ as diameter in Q; prove that $\frac{PN}{QN}=\frac{CB}{CA}$.

120. PCP', DCD' are two conjugate diameters of a conic; QQ' is a chord parallel to DD' cutting PP' at N; a line through N parallel to PD cuts CD at R; prove that $QN^2=DR.RD'$.

121. PP' is a fixed chord of a parabola, perpendicular to the principal axis: a variable diameter cuts the curve in Q, PP' in N, and the circle on PP' as diameter in R; prove that $\frac{RN^2}{QN}$ is constant.

122. T is the pole of a chord PQ of a conic; a chord HK parallel to TP meets PQ, TQ at V, R; prove that $RV^2=RH.RK$.

123. Prove that the ratio of two focal chords of a conic is equal to that of the squares of the semi-diameters parallel to them. [Use Ex. 95, Chapter V.]

124. P, Q, R, are three points on a parabola; the diameters through Q, R meet the tangent at P in H, K; prove that $\frac{HP^2}{PK^2}=\frac{HQ}{KR}$.

125. If a conic and a circle intersect at P, Q, R, S, prove that the diameters of the conic parallel to PQ and RS are equal; and that PQ and RS are equally inclined to either principal axis.

126. QVQ' is a double ordinate to the diameter PP' of a conic; prove that $\frac{PQ}{PQ'}$ is equal to the ratio of the parallel semi-diameters.

127. PQ, PR are two chords of a conic, equally inclined to the tangent at P; CH, CK are the semi-diameters parallel to PQ, PR; prove that $\frac{PQ}{PR}=\frac{CH^2}{CK^2}$.

128. $P\,Q$ are two points on a conic; the ordinate PN to a principal axis AA' meets AQ, $A'Q$ at H, K; prove that $PN^2=HN.KN$.

129. AB is a fixed diameter of a circle; Q is the pole of a variable chord AP; prove that the locus of the meet of AP, BQ is a conic.

130. T is a point on the tangent at a point P of a parabola; any line through T cuts the parabola at Q, R and the diameter through P in H; prove that $TQ.TR=TH^2$.

131. Two conics s_1, s_2 cut at A, B, C, D; x_1, y_1 and x_2, y_2 are the lengths of the pairs of diameters of s_1 and s_2 parallel to AB, CD; prove that $x_1\,y_2=x_2\,y_1$.

132. Two conics s_1, s_2 cut at A, B, C, D; from any point P on AB, two lines PH_1K_1, PH_2K_2 are drawn cutting s_1, s_2 at H_1, K_1 and H_2, K_2 respectively; prove that the points C, D, H_1, K_1, H_2, K_2 lie on a conic.

133. X is the foot of the perpendicular from the focus S of a parabola to the directrix; PSQ is a chord; a line through X, parallel to PQ, cuts the curve at H, K; prove that $XH.XK=PS.SQ$.

134. A chord PQ of a conic subtends equal angles at the extremities of a chord RS; prove that it subtends equal angles at the extremities of any chord parallel to RS, if the meet of PQ, RS lies outside the conic.

135. The circle of curvature at a point P of a conic cuts a chord PK of the conic at L; x, y are the lengths of the focal chords parallel to PK and the tangent at P; prove that $\frac{PK}{PL}=\frac{x}{y}$.

CHAPTER VII.

RECIPROCATION.

THE Principle of Duality springs from a recognition of the fact that a curve may be regarded, both as the path of a moving point, and as the envelope of a moving line. The latter less obvious idea appears to be due to De Beaune (1601-1652), one of the many commentators on the work of Descartes. The "Horologium" of Huygens (1629-1695), the inventor of the watch and the earliest writer on the undulatory theory of light, contains some mention of the properties of the evolute (*i.e.* the envelope of the normals) of a parabola and cycloid: while some optical applications to caustics are due to Tschirnhausen (1631-1708). A systematic treatment of envelopes was given in 1692 by Leibnitz, who shares with Newton the honour of the discovery of the infinitesimal calculus. The advantage derived from coordinating these two conceptions was first pointed out by Brianchon (1806); while the principle itself was stated very clearly by Gergonne in 1825, to whom the notion of the **class** of a curve is due. Poncelet, however, was mainly responsible for the important and extensive developments, which have made the theory one of the dominant influences in modern geometry. Some applications to metrical properties were given by Chasles and Salmon: and the complementary analytical treatment. by means of line coordinates, was supplied by Möbius (1790-1868), and Plücker (1829).

Definition.

A, B, C, ... ; l, m, n, ... are any given system of points and lines in a plane, and Σ is any given conic in the same plane. a, b, c, ... ; L, M, N, ... are the polars and poles of A, B, C, ... ; l, m, n, ... w.r.t. the conic Σ. Then the system a, b, c, ... ; L, M, N, ... is called the **reciprocal** of the given system w.r.t. the **base-conic** Σ.

It follows at once from the definition that the first system is also the reciprocal of the second w.r.t. Σ. Moreover the correspondence thus established between the two figures is (1, 1). For, to any point of the first system corresponds one and only one line of the second; and to any line of the first system corresponds one and only one point of the second. A brief introductory account of the Principle of Duality will be found in Chapter IX. of the first part of this treatise.

Notation.

Small letters will be used to denote lines, and capital letters to denote points. Correspondence in the two figures will be indicated by the use of the same letter: *e.g.* the line l in the first system corresponds to the point L in the second.

To avoid confusion, the original figure and the reciprocal figure will usually be drawn out separately.

THEOREM 95.

(1) If the reciprocals of A, B meet at P, then P is the reciprocal of AB.	If the reciprocals of a, b lie on p, then p is the reciprocal of ab.
(2) If A, B, C, D, ... lie on l, then a, b, c, d, ... pass through L.	If a, b, c, d, ... pass through L, then A, B, C, D ... lie on l.

The proof is left to the reader.

THEOREM 96.

(1) If $\{ABCD\}$ is harmonic, then $\{abcd\}$ is harmonic.	If $\{abcd\}$ is harmonic, then $\{ABCD\}$ is harmonic.
(2) The cross ratio of $\{ABCD\}$ equals the cross ratio of $\{abcd\}$.	The cross ratio of $\{abcd\}$ equals the cross ratio of $\{ABCD\}$.

Definition.

A moving point P traces out a curve S_1, and p is the polar of P w.r.t. the base conic Σ, then the curve S_2, generated or enveloped by p, is called the reciprocal of S_1 w.r.t. Σ.

It is of fundamental importance to show that, if the curve S_2 is the reciprocal of the curve S_1, then S_1 is also the reciprocal of S_2, as previously defined; that is to say, if we take a moving tangent t of S_1, then the locus of its pole T w.r.t. Σ is the same curve S_2 as is obtained by taking a moving point P of S_1 and forming the envelope of the polar p of P w.r.t. Σ. Or in other words, whether we regard S_1 as the locus of a moving point, or the envelope of a moving line, the same reciprocal curve is obtained.

THEOREM 97.

If the curve S_2 is the reciprocal of S_1 w.r.t. Σ, then S_1 is the reciprocal of S_2 w.r.t. Σ.

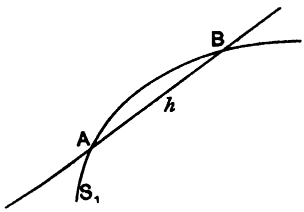

FIG. 86.

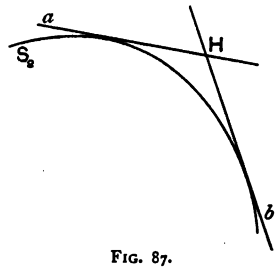

FIG. 87.

Let h be the join of two adjacent points A, B on S_1.

Then, by Theorem 53, the reciprocal H of h is the meet of two adjacent tangents a, b to S_2.

In the limit, when B tends to coincide with A, h becomes a tangent to S_1 and H becomes a point on S_2; so that the reciprocal of a tangent to S_1 is a point on S_2.

S_2 can therefore be generated from S_1 by regarding S_1 as an envelope. Q.E.D.

THEOREM 98.

The reciprocal of a conic S_1 w.r.t. a base conic Σ is another conic S_2.

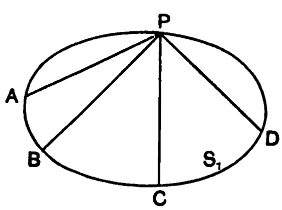

FIG. 88.

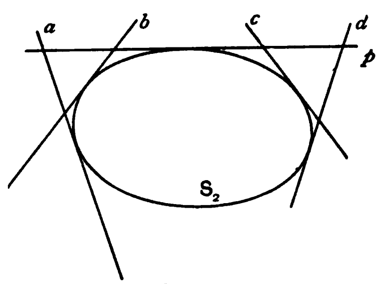

FIG. 89.

Take four fixed points A, B, C, D on S_1: and let P be a variable point on S_1.

Let a, b, c, d, p be the reciprocals of A, B, C, D, P.

Then $\qquad p\{abcd\} = P\{ABCD\}$ [Th. 55.]

$\qquad\qquad = \text{constant}$; [Th. 57.]

$\therefore$ p envelopes a conic; [Th. 75.]

$\therefore$ the reciprocal of S_1 is a conic. Q.E.D.

By assuming (from analysis), see Theorem 51, that every curve of the second degree, *or* of the second class is a conic, it is easy to give another proof of Theorem 98.

Any straight line cuts the conic S, at two and only two points. Therefore from every point, two and only two tangents can be drawn to the reciprocal S_2 of S_1. Therefore S_2 is a curve of the second class, and is therefore a conic. Q.E.D.

In the following theorems, S_2 will denote the reciprocal of the conic S_1 w.r.t. the base-conic Σ.

THEOREM 99.

(1) If a line a meets S_1 at P, Q, then the tangents from A to S_2 are p, q; and conversely.

(2) If the join of A, B touches S_1, then the meet of a, b lies on S_2; and conversely.

(3) If A, B, C, D are the common points of S_1, S'_1, then a, b, c, d are the common tangents of S_2, S_2; and conversely.

(4) If S_1, S_1 touch at a point A, then a is a common tangent to S_2, S'_2 at their point of contact.

The proof is left to the reader.

THEOREM 100.

If a is the polar of B w.r.t. S_1, then A is the pole of b w.r.t. S_2.

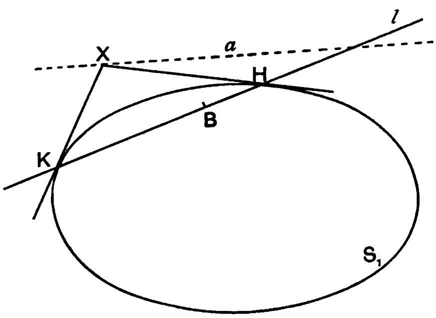

Fig. 90.

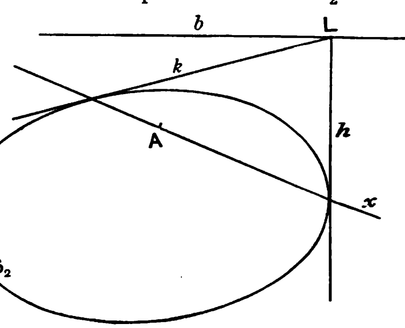

Fig. 91.

Draw through B, a variable line l, cutting S_1 at H, K; let the tangents at H, K meet at X; then X traces out a.

In the reciprocal figure, L is a variable point on b, the tangents from which to S_2 are h, k; and the join of the points of contact of these tangents is x.

Then, by definition, the reciprocal of a is the envelope of x.

Now the pole of x lies on b;

$\therefore$ the pole of b lies on x;

$\therefore$ x envelopes the pole of b;

$\therefore$ the reciprocal of a, viz. A, is the pole of b.

Q.E.D.

THEOREM 101.

(1) If P, Q are conjugate points w.r.t. S_1, then p, q are conjugate lines w.r.t. S_2; and conversely.

(2) If PQR is a self-conjugate triangle w.r.t. S_1, then pqr is a self-conjugate triangle w.r.t. S_2.

The proof is left to the reader.

THEOREM 102.

(1) If O is the centre of the base-conic Σ, the centre of S_1 reciprocates into the polar of O w.r.t. S_2, and the polar of O w.r.t. S_1 reciprocates into the centre of S_2.

(2) The reciprocal of O is the line at infinity; and conversely.

(3) The asymptotes of S_1 reciprocate into the meets of S_2 with the polar of O w.r.t. S_2.

(4) S_2 is an ellipse, parabola, or hyperbola, according as O lies inside, on, or outside S_1.

The proof is left to the reader.

1. Prove Theorem 95.

2. Prove Theorem 96.

3. Prove Theorem 99.

4. Prove Theorem 101.

5. Prove Theorem 102.

6. If two conics touch each other, prove that their reciprocals touch each other.

7. What is the reciprocal of a common chord of two conics?

8. What is the reciprocal of a common tangent of two conics?

9. Two conics cut at P; what is the reciprocal of the tangents at P to the conics?

10. What is the reciprocal of two parallel tangents to a conic?

11. What is the reciprocal of a pair of conjugate diameters of a conic?

12. What is the reciprocal of two parallel chords of a conic?

13. P, Q are two points on a tangent h to a conic S_1; the other tangents from P, Q to S_1 meet at R; construct the reciprocal figure.

14. PQR is a triangle, self-conjugate to a conic S_1, and circumscribing a conic S'_1; what is the reciprocal figure?

Attention has already been directed to the fact that all descriptive properties occur in pairs. The application of this principle to the geometry of the conic depends on the fundamental property of Theorem 98. The following example will illustrate the process.

EXAMPLE.

To obtain the dual of the following theorem: a variable conic touches four fixed lines, then the locus of the poles of a given line is a fixed line.

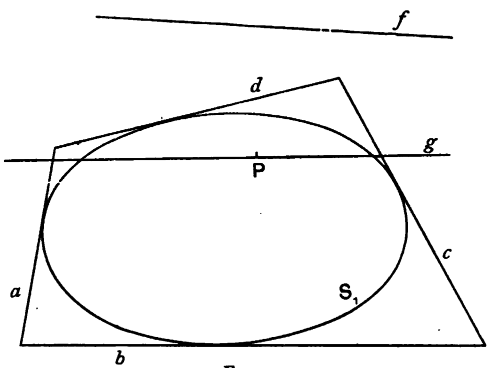

FIG. 92.

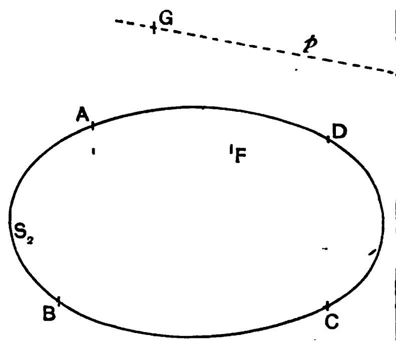

FIG. 93.

Let the variable conic S_1 touch four fixed lines a, b, c, d.

Let f be the given line, and P the pole of f w.r.t. S_1.

Then the reciprocal conic S_2 passes through four fixed points A, B, C, D, and p is the variable polar of the fixed point F w.r.t. the variable conic S_2.

Now P traces out a fixed line g.

Therefore p passes through a fixed point G.

Hence we have the following: a variable conic passes through four fixed points, then the polars of a given point pass through a fixed point. Q.E.D.

At first it is worth while for the reader to draw the reciprocal figure of the given figure, bit by bit, and so gradually work up to the enunciation of the dual theorem. But after a little practice, it

becomes clear that the process may be made quite mechanically, without the necessity of referring at each stage to the figure.

In the given enunciation, it is merely necessary to make a certain set of verbal changes. These are given here in two columns. When an element in *either* column occurs, it must be replaced by the corresponding element in the other.

Point	Line
Concurrent	Collinear
quadrangle	quadrilateral
range	pencil
base of range	vertex of pencil
join of two points	meet of two lines
lie on	pass through
locus	envelope
degree	class
point on a conic	tangent to a conic
meets of a line with a conic	tangents from a point to a conic
pole	polar

Write down, without proof, the dual theorems of Ex. 15-40.

15. The sides of a variable triangle pass through fixed non-collinear points, and two of the vertices move on fixed lines, then the locus of the third vertex is a conic.

If however the fixed points are collinear, then the locus of the third vertex is a straight line.

16. If a conic touches the sides BC, CA, AB of a triangle at P, Q, R, then AP, BQ, CR are concurrent.

17. If a hexagon is inscribed in a conic, the meets of opposite sides are collinear.

18. A variable triangle is inscribed in a fixed conic; if two of its sides pass through fixed points, then the third side touches a conic having double contact with the given conic.

19. Four conics can be drawn to touch three given lines, and to pass through two fixed points.

20. Two conics can be drawn to pass through four fixed points and to touch a given line.

21. Two conics touch at A and cut again at B, C; any line through A cuts the conics again at P, Q; then the tangents at P, Q meet on BC.

22. If three conics have two common points, then the six meets of their common tangents lie three by three on four straight lines.

23. If a quadrangle is inscribed in a conic, its diagonal point triangle is self-conjugate w.r.t. the conic.

24. If a variable conic passes through four fixed points, then the locus of the poles of a given line is a conic.

25. If the polars of P, Q, R w.r.t. a conic meet QR, RP, PQ at P', Q', R', then P', Q', R' are collinear.

26. A, B are two fixed points; P is a variable point such that the tangents from P to a fixed conic are harmonically conjugate to PA, PB; then the locus of P is a conic.

27. From a variable point P, on a common chord of two fixed conics, tangents are drawn, meeting the conics at H, K; then HK passes through one of two fixed points.

28. Two conics S_1, S_2 have double contact at A, B; PQ is a variable chord of S_1 and touches S_2; if H is any point on S_1, then $H\{APBQ\}$ is constant.

29. A, B, C, P, Q, R are six fixed points on a conic; two conics are drawn to have double contact with each other and to circumscribe the triangles ABC, PQR respectively; then their chord of contact envelopes a conic.

30. If two triangles are self-conjugate w.r.t. a conic, then their six vertices lie on a conic.

31. A, B, C, D are four fixed points on a conic; a variable line through A cuts BC, CD, DB at D', B', C'; then $\{AB'C'D'\}$ is constant.

32. A system of conics have double contact at A, B; C is any other fixed point; AC, BC cut any one of the conics at P, Q; then PQ passes through a fixed point.

33. The tangents from a variable point P to a fixed conic are harmonically conjugate to the tangents from P to a second fixed conic; then the locus of P is a conic.

34. A variable conic touches two fixed lines and passes through two fixed points; then the locus of the pole of the join of the fixed points is two lines.

35. P, Q are conjugate points w.r.t. a given conic; the tangents from Q to the conic meet a fixed line at H, K. If P is fixed, then PH, QK meet on a fixed line.

36. S_1, S_2, S_3 are three conics inscribed in the same quadrilateral; P is a point of intersection of S_1 and S_2; then the tangents at P to S_1, S_2 are harmonically conjugate to the tangents from P to S_3.

37. A variable conic passes through two fixed points and touches two fixed lines; then the chord of contact passes through one of two fixed points.

38. If two conics touch at A, and if two lines AP, AR cut the conics at P, R and Q, S, then the chords PR, QS meet on the common chord of the two conics.

39. From two points on a common chord of two conics, tangents are drawn, one to each conic; then the diagonals of the quadrilateral thus formed, each pass through a meet of the common tangents.

40. Two conics have each double contact with a third conic; then the chords of contact and a pair of common chords are concurrent, and form a harmonic pencil.

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41. Reciprocate the Example on page 176, taking the centre of the base-conic at a corner of the quadrilateral.

42. Reciprocate Ex. 38, taking the centre of the base-conic at A; and deduce a special case when APQ, ARS coincide.

43. Reciprocate w.r.t. a base-conic, centre A: the locus of the centre of a conic touching three fixed lines, and passing through a fixed point A, is a conic.

44. If the conic S_3 is the reciprocal of the conic S_1 w.r.t. S_2, prove that S_1, S_2, S_3 have a common self-conjugate triangle.

45. Prove by reciprocation and projection: a variable conic is inscribed in a given triangle ABC, and has A, D as conjugate points, D being a fixed point; prove that the conic touches another fixed line.

46. Prove by reciprocation and projection, Ex. 36.

POINT RECIPROCATION.

We proceed to consider a special case of reciprocation, arising when the base-conic is a circle. If O is the centre of the base-circle and k its radius, the process is called reciprocating w.r.t. O, and k is called the radius of reciprocation. In general, the value of k is immaterial; for it is easy to see that an alteration in the value of k merely changes the figure into a homothetic figure. The utility of point-reciprocation is due to the fact that the reciprocal of a conic w.r.t. its focus is a circle. Three proofs of this will be given. The first depends on an elementary property found in all text-books on geometrical conics; the second follows logically from the present development of the subject; and the third is analytical.

THEOREM 103.

The reciprocal of a conic w.r.t. one of its foci is a circle.

First Method.

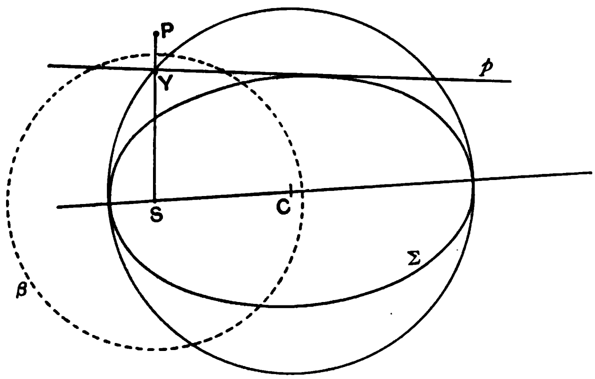

FIG. 94.

Let S be the focus, and p a variable tangent to the conic; Y is the foot of the perpendicular from S to p. SY is produced to P, so that $SY.SP = k^2$, where k is the radius of reciprocation.

Then P is the polar of p w.r.t. the circle, centre S, radius k;

$\therefore$ P is the reciprocal of p.

Now Y moves on the auxiliary circle of the conic, or, if the conic is a parabola, on the tangent at the vertex.

But P traces out the inverse curve of the locus of Y, w.r.t. S; therefore the locus of P is a circle, which passes through S if the conic is a parabola. Q.E.D.

Second Method.

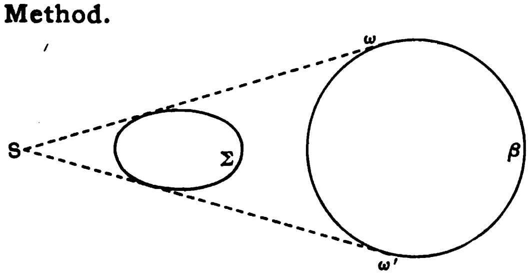

FIG. 95.

Let S be the focus of Σ, and denote the circular points at infinity by ω, ω'.

Then $S\omega$, $S\omega'$ are tangents to Σ.

Now $S\omega$ touches the base-circle β, centre S, at ω;

$\therefore$ the reciprocal of $S\omega$ is ω;

$\therefore$ the reciprocal of Σ passes through ω and similarly through ω'.

But the reciprocal of a conic is a conic. [Th. 98.]

Therefore the reciprocal of Σ is a conic through ω, ω'; and is therefore a circle. Q.E.D.

Third Method.

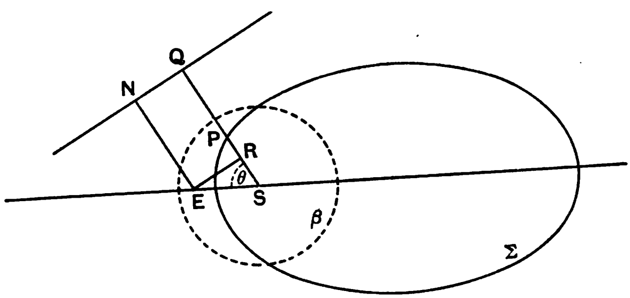

FIG. 96.

Take the focus S of Σ as origin and the base-line along the major axis.

Then the equation of Σ is $\frac{l}{r} = 1 + e\cos\theta$, where e is the eccentricity, and l the semi-latus rectum.

Let k = radius of reciprocation: P is any point on Σ.

Produce SP to Q so that $SP \cdot SQ = k^2$; and draw QN perpendicular to SQ. Then QN is the reciprocal of P.

Let (r, θ) be the coordinates of P.

Since $$r = \frac{k^2}{SQ}, \quad \frac{l \cdot SQ}{k^2} = 1 + e\cos\theta;$$

$$\therefore \quad SQ = \frac{k^2}{l} + \frac{ek^2}{l}\cos\theta; \text{ or } SQ - \frac{ek^2}{l} \cdot \cos\theta = \frac{k^2}{l}.$$

Take a point E on the major axis, such that $ES = \frac{ek^2}{l}$.

Draw EN, ER perpendicular to QN, QS.

Then $$EN = RQ = SQ - SR = SQ - \frac{ek^2}{l} \cdot \cos\theta = \frac{k^2}{l}.$$

Therefore QN envelopes a circle, centre E, radius $\frac{k^2}{l}$, where E is at a distance $\frac{ek^2}{l}$ from the focus. Q.E.D.

THEOREM 104.

The reciprocal of a circle w.r.t. any point S is a conic having one focus at S; and if S lies on the circle, then the conic is a parabola.

The proof is left to the reader.

[It is only necessary to reverse the order of the argument used in either of the methods given for Theorem 103.]

General Properties.

For the sake of brevity, a definite notation is adopted in the following pages.

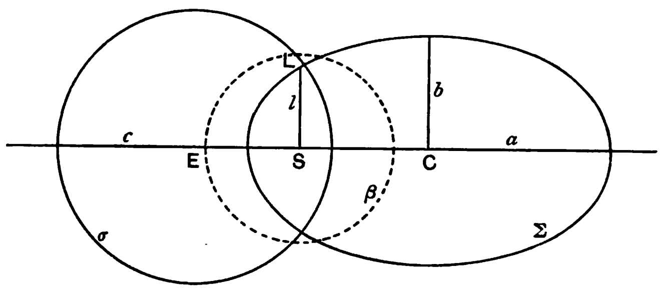

FIG. 97.

The circle σ and the conic Σ are reciprocally related to each other by the base-circle β, whose centre is the focus S of Σ and whose radius is k; E is the centre, and c is the radius of σ; C is the centre and a, b are the semi-axes of Σ; and l is the length of the semi-latus rectum SL of Σ.

THEOREM 105.

(1) The point S and the line at infinity are reciprocals.

(2) The points at infinity on Σ are reciprocal to the tangents from S to σ.

(3) The asymptotes of Σ are reciprocal to the points of contact with σ of the tangents from S to σ.

(4) Σ is an ellipse, parabola, or hyperbola according as S lies inside, on, or outside σ.

The proof is left to the reader. [Use Theorem 100.]

THEOREM 106.

(1) The point C and the polar of S w.r.t. σ are reciprocals.

(2) The directrix of Σ and the point E are reciprocals.

The proof is left to the reader.

[Note that C is the pole of the line at infinity w.r.t. Σ; and that the directrix of Σ is the polar of S; and use Theorem 100.]

47. Use the second method of proof of Theorem 103 to show that the reciprocal of a parabola, focus S, w.r.t. S, is a circle through S.

48. Prove Theorem 104.

49. Prove Theorem 105.

50. Prove Theorem 106.

51. Prove that a system of parallel lines reciprocate into a system of points, collinear with the origin.

52. What is the reciprocal of the extremities of a double ordinate of a conic w.r.t. a focus S?

53. ABC is a triangle; SA meets BC at P; what is the reciprocal of P w.r.t. S?

54. What is the reciprocal of a quadrangle w.r.t. a diagonal point?

55. A circle is reciprocated w.r.t. a point outside it; what are the two parts of the circle which correspond to the two branches of the reciprocal hyperbola?

56. A conic is reciprocated w.r.t. a focus; what is the reciprocal of a pair of conjugate diameters?

57. A line parallel to the base BC of a triangle ABC meets AB, AC at D, E; what is the reciprocal figure w.r.t. A?

58. What is the reciprocal of two circles w.r.t. a centre of similitude?

59. Reciprocate w.r.t. A: two circles touch at A; a line PAQ meets the circles at P, Q; then the tangents at P, Q are parallel.

60. Given four points S, A, B, C, prove that in general, four conics can be drawn through A, B, C having S as focus; and that three of the conics are hyperbolas with A, B, C not all on the same branch; while the remaining conic may be an ellipse, parabola, or hyperbola having A, B, C on the same branch.

61. Two conics touch at A, cut at B, C, and have a common focus S; if S lies on BC, what is the reciprocal figure w.r.t. S?

62. O is the circumcentre of the triangle ABC; P, Q, R are the poles of BC, CA, AB w.r.t. another circle, centre O; prove that O is the incentre or an excentre of the triangle PQR.

63. A conic touches two fixed lines and has a given focus; find the locus of its centre.

64. Three conics have a common focus; prove that the meets of the common tangents of the conics, taken in pairs, are collinear.

65. If two conics have a common focus, prove that a pair of common chords will pass through the meet of the directrices corresponding to that focus.

66. A variable conic touches two fixed lines and has a given focus; prove that its directrix passes through one of two fixed points.

67. Reciprocate (i) w.r.t. C, (ii) w.r.t. any point: the locus of the centre of a variable circle, which touches a fixed circle, centre C, and also a fixed straight line, is a parabola, focus C.

68. Three conics have a common focus, and each pair intersect at four real points: prove that their common chords form four groups of three concurrent lines, each of the three being a common chord of a different pair of conics.

69. A variable conic Σ touches internally each of two fixed conics at variable points P, Q. If the three conics have a common focus, prove that the pole of PQ w.r.t. Σ lies on a fixed line.

70. Reciprocate w.r.t. (i) A, (ii) any point: S_1, S_2 are two fixed circles, centres A, B; a variable circle Σ touches S_1, S_2; then the locus of its centre is two confocal conics, A, B being the foci.

71. Two parabolas have a common focus. From any point on their common tangent, two other tangents are drawn to the parabolas. Prove that another parabola, with the same focus, can be drawn to touch the last two tangents, and the line joining their points of contact.

72. PQ is a double ordinate of a parabola; if the line joining P to the foot of the perpendicular from the focus S to the directrix cuts the curve again at P', prove that $P'Q$ passes through S.

73. A variable parabola touches a fixed conic and has its focus at one of the foci of the given conic; prove that its directrix touches a fixed circle.

74. P is a point on a parabola, focus S; SP and the tangent at P meet the directrix in M, M'. The joins of M, M' to the vertex meet the curve again at Q, Q'; prove that QQ' is a focal chord.

75. PQ is a variable chord of the auxiliary circle of an ellipse, and touches the ellipse; prove that the locus of the pole of PQ w.r.t. the auxiliary circle is a similar ellipse. [The auxiliary circle is the circle having the major axis as diameter.]

76. Reciprocate w.r.t. any point: two chords PQ, PR of a conic are parallel to a pair of conjugate diameters, if QR is a diameter.

77. If two of the six meets of four tangents to a parabola lie on the axis, prove that the remaining four are equidistant from the focus.

THEOREM 107.

(1) With the previous notation, $c=\frac{k^2}{l}$.

(2) If the perpendicular through S to ES meets σ at H, then $b=\frac{k^2}{SH}$.

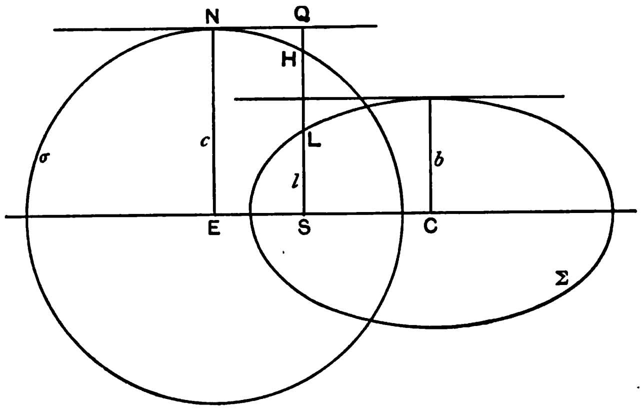

FIG. 98.

(1) Produce SL to Q so that $SL \,.\, SQ = k^2$; and draw QN perpendicular to SQ; QN is therefore by definition a tangent to σ; let N be the point of contact, so that $E\hat{N}Q = 90°$.

But QN is parallel to ES, and EN is parallel to QS.

$$\therefore \; c = \text{distance of } E \text{ from } QN = SQ = \frac{k^2}{SL} = \frac{k^2}{l}. \qquad \text{Q.E.D.}$$

(2) H reciprocates into a tangent to Σ, parallel to the major axis. But the distance of this line from S is $\frac{k^2}{SH}$.

$$\therefore \; b = \frac{k^2}{SH}. \qquad \text{Q.E.D.}$$

78. With the usual notation, prove that $a=\frac{k^2}{c^2 \sim ES^2}\cdot c$.

79. Prove that the eccentricity of Σ equals $\frac{ES}{c}$.

80. What are the reciprocals of the extremities of the axes of Σ?

81. What is the reciprocal of the second focus of Σ?

82. What is the reciprocal of the foot of the perpendicular from S to the corresponding directrix?

83. What is the reciprocal of two conjugate points on the directrix, corresponding to S?

84. What is the reciprocal of the minor axis of Σ?

85. What is the reciprocal of a system of coaxal circles w.r.t. a point on their radical axis?

86. Prove that a system of equal circles reciprocate into a system of conics, having a common focus and equal latera recta.

87. Prove that the major axis of an ellipse, which is the reciprocal of the circumcircle of a triangle w.r.t. its incircle, is equal to the inradius.

88. Reciprocate w.r.t. O: through a fixed point O, a variable line is drawn, cutting a fixed circle at P, Q; then $OP \cdot OQ$ is constant.

89. Reciprocate w.r.t. O: OA is a fixed diameter of a circle OPA; Y is the foot of the perpendicular from O to the tangent at P; then $OP^2 = OY \cdot OA$.

90. Two coaxial parabolas have a common focus S; prove that the sum of their latera recta equals $4SP$, where P is one of their common points.

91. Two conics with the same focus and directrix are such that triangles can be inscribed in one and circumscribed to the other; prove that the eccentricity of one is twice that of the other.

92. A variable conic of given focus and latus rectum passes through a fixed point; prove that the directrix passes through one of two fixed points.

93. Two conics, having a common focus S, intersect at two and only two real points P, Q; H, K are the poles of PQ w.r.t. the conics; prove that H, S, K are collinear.

94. I is the incentre and r the inradius of an equilateral triangle ABC; a hyperbola is drawn to circumscribe the triangle and to have one focus at I; prove that its eccentricity $= \frac{4}{3}$, and that its latus rectum $= \frac{4}{3}r$.

95. A variable conic has a given focus S, and touches two fixed lines OA, OB; prove that its minor axis envelopes a parabola, of which S and a line through O are focus and directrix.

96. Prove that four conics can be drawn to circumscribe a given triangle, and to have a given point S as focus; and that there exists another conic Σ, having S as focus, and touching these four conics; further, if σ is the conic inscribed in the given triangle, with S as focus, the latus rectum of Σ is double that of σ.

97. Reciprocate w.r.t. P: T is the pole of a chord QR of a circle PQR; PL, PM, PN are the perpendiculars from P to QR, TQ, TR; then $PL^2=PM.PN$.

98. Reciprocate w.r.t. O: P is a variable point on a fixed circle Σ; O is a fixed point; a point P' is taken on OP such that $OP.OP'$ is constant; then the locus of P' is a circle, unless O lies on Σ, in which case it is a straight line.

99. PQR is a triangle circumscribing a parabola, focus S; if R' is the point of contact of PQ, prove that $SR.SR'=SP.SQ$.

100. A focal chord PSQ cuts the corresponding directrix at R; prove that $\{PQ;RS\}$ is harmonic.

101. Reciprocate w.r.t. O: the envelope of the polars of a fixed point O, w.r.t. a system of equal circles passing through a fixed point, is a conic.

102. O is the circumcentre of the triangle ABC; prove that the major axis of the conic inscribed in ABC with one focus at O is equal to OA.

103. PQ is a variable chord of an ellipse, eccentricity e, subtending a right angle at the focus S; prove that the locus of the pole of PQ is a hyperbola, parabola, or ellipse, according as $e>, =, <\frac{1}{\sqrt{2}}$.

104. Reciprocate w.r.t. any point: a fixed circle cuts a variable circle belonging to a given coaxal system at P, Q; then PQ passes through a fixed point situated on the radical axis of the system.

105. Reciprocate w.r.t. any point: A, B, C are the centres of three circles, each of which touches the other two; if A', B', C', are the corresponding points of contact, then AA', BB', CC', are concurrent.

106. H is the orthocentre of a triangle ABC; S_1 is a conic having H as focus and AB as directrix; S_2 is a conic having H as focus and AC as directrix; if S_1, S_2, touch BC, prove that their minor axes are equal.

107. The four conics circumscribing the triangle ABC, and having a point S as focus, cut, two by two, at the six points D_{12}, $D_{13}, \ldots$; prove that each of the three conics having S as focus, one side of the triangle ABC as directrix, and passing through the opposite vertex contains two of the points D.

108. A, A'; B, B'; C, C' are the pairs of opposite vertices of a quadrilateral circumscribing a parabola, focus S; prove that $SA.SA'=SB.SB'=SC.SC'$.

109. The sides of a triangle ABC touch a parabola at P, Q, R; if S is the focus, prove that $SA.SB.SC=SP.SQ.SR$.

THEOREM 108.

If the lines p, q are the reciprocals of the points P, Q w.r.t. a point S, then the angle PSQ is equal to the angle between p, q.

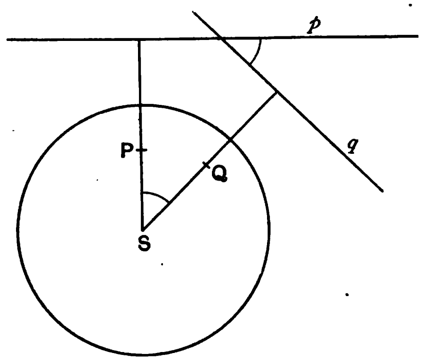

FIG. 99.

SP, SQ are, by definition, perpendicular to p, q.

$\therefore$ $P\hat{S}Q$ = angle between p, q. Q.E.D.

110. A is the foot of the perpendicular from O to a line BC; what is the reciprocal figure w.r.t. O?

111. What is the reciprocal of a triangle w.r.t. its orthocentre?

112. A line parallel to the hypotenuse BC of the right-angled triangle ABC cuts AB, AC at P, Q; what is the reciprocal figure w.r.t. A?

113. What is the reciprocal of two orthogonal circles (i) w.r.t. any point, (ii) w.r.t. a point of intersection?

114. What is the reciprocal of a parabola w.r.t. a point on the directrix?

115. What is the reciprocal of a conic w.r.t. a point on the director circle?

116. O is a point on the circumcircle of the triangle ABC; $A'B'C'$ is the reciprocal of ABC w.r.t. O; prove that O, A', B', C' are concyclic.

117. A', B', C' are points on the sides BC, CA, AB of a triangle, such that the circles on AA', BB', CC' as diameters have a common point; prove that A', B', C' are collinear.

118. An asymptote CE of a hyperbola meets a directrix at E; S is the corresponding focus; prove that $C\hat{E}S = 90°$.

119. Two parabolas have a common focus S; prove that their common tangent subtends at S an angle equal to the angle between their axes.

120. Two chords PR, QR of a conic meet the directrix at L, M; S is the corresponding focus; prove that $L\hat{S}M = 90° \pm \frac{1}{2}P\hat{S}Q$.

121. Two sides of a triangle are given in position, and the third side subtends a constant angle at a fixed point; find its envelope.

122. Reciprocate w.r.t. O: two tangents to a conic from a point T meet a fixed line at P, Q; if PQ subtends a right angle at a fixed point O, the locus of T is a conic.

123. Find a point O such that the reciprocal of a given triangle w.r.t. O is a similar triangle.

124. Find the envelope of a variable chord of a conic, subtending a constant angle at the focus. Find also the locus of its pole.

125. PQR is a variable triangle circumscribing a conic, focus S; if $P\hat{S}Q$ is constant, find the locus of R.

126. If two parabolas have a common focus, prove that their common chord bisects the angle formed by their directrices.

127. T is a variable point on a fixed tangent to an ellipse; Q is a point on the other tangent from T to the ellipse, such that TQ subtends a right angle at the focus; prove that the locus of Q is a straight line.

128. T is the pole of a chord PQ of a conic, focus S; PQ meets the directrix at R; prove that SR, ST are the bisectors of the angle PSQ.

129. Two parabolas, having a common focus S, cut at P, Q; prove that their common tangent is parallel to a bisector of $P\hat{S}Q$.

130. From any point P on a common tangent to two ellipses, which have a common focus S, tangents are drawn to cut the other common tangent at Q, R; prove that $Q\hat{S}R$ is constant.

131. T is a point on the tangent at a variable point P on a parabola: if $S\hat{T}P$ is constant, find the locus of T.

132. If two parabolas have a common focus, and their axes in opposite directions, prove that they cut orthogonally.

133. Find the locus of a point from which tangents to a fixed parabola are inclined at a constant angle.

134. Reciprocate w.r.t. A: T is the pole of a chord AB of a circle, then $T\hat{A}B = T\hat{B}A$.

135. Reciprocate (i) w.r.t. the focus, (ii) w.r.t. the vertex: if a chord PQ of a parabola subtends a right angle at the vertex, then the locus of its pole is a straight line, parallel to the directrix.

136. A conic, inscribed in the triangle ABC, touches BC at A'; if S is a focus, prove that $B\hat{S}A' + A\hat{S}C = 180°$.

137. Reciprocate w.r.t. any point: the locus of points from which tangents to an ellipse are at right angles is a circle, concentric with the ellipse.

138. O is a fixed point on a conic; PQ is a variable chord. If $P\hat{O}Q = 90°$, prove that PQ passes through a fixed point.

139. Prove that the opposite sides of a quadrilateral, circumscribing an ellipse, subtend supplementary angles at a focus.

140. If two parabolas have a common focus, prove that the line joining the focus to the meet of the directrices is perpendicular to the common tangent.

141. Two parabolas have a common focus, and their axes in opposite directions; a straight line parallel to the axis meets them in P, P'; prove that the tangents at P, P' intersect on the common chord and subtend equal angles at the focus.

142. From any point on the common tangents of two parabolas, having a common focus, two other tangents are drawn; prove that they make with each other a constant angle.

143. Reciprocate w.r.t. B: two circles cut at A, B; a variable line through B cuts the circles again at P, Q; then $P\hat{A}Q$ is constant.

144. Reciprocate w.r.t. O: AB is a diameter of a circle $AOPB$; then $A\hat{P}B = 90°$.

145. Prove that the locus of the foot of the porpendicular from the focus of a parabola to a variable tangent is a straight line.

146. $ABCD$ is a quadrilateral circumscribing a conic; the diagonals AC, BD meet at a focus; prove that AC is perpendicular to BD, and that the directrix is the third diagonal.

147. Reciprocate w.r.t. the focus, Steiner's theorem: the orthocentre of a triangle formed by three tangents to a parabola lies on the directrix.

148. S is a focus, C the centre, and CE an asymptote of an hyperbola; the tangent at a variable point P on the curve cuts the directrix in T; Q is a point on the directrix such that $Q\hat{S}T = E\hat{C}S$; prove that PQ envelopes a parabola.

149. The tangents from a variable point P to a conic meet the directrix in conjugate points; find the locus of P.

150. Reciprocate w.r.t. any point: PQ is a chord of a circle; then the angle between PQ and the tangent at P is equal to the angle in the alternate segment.

151. Reciprocate w.r.t. any point: P, P'; Q, Q' are two pairs of inverse points w.r.t. a circle Σ; then P, P', Q, Q' lie on a circle, orthogonal to Σ.

152. PQ, PR are two chords of a rectangular hyperbola; if $Q\hat{P}R=90°$, prove that the tangent at P is perpendicular to QR.

153. The tangent at any point P of a hyperbola cuts an asymptote at T; and a line through P parallel to that asymptote cuts the directrix at K; prove that KT subtends a right angle at the corresponding focus.

154. A triangle PQR circumscribes a parabola, focus S; lines are drawn through P, Q, R making equal angles with SP, SQ, SR respectively; prove that they are concurrent.

155. SP is drawn through a focus S of a hyperbola, parallel to an asymptote, and cuts the curve at P; prove that the tangent at P meets the other asymptote on the latus rectum produced.

156. Two vertices A, B of the triangle ABC are fixed. The bisector of the angle BAC meets BC at a point on a fixed line. Find the locus of C.

157. T is the pole of a chord PQ of a parabola, focus S; prove that the triangles TPS, QTS are similar.

158. Reciprocate w.r.t. S: PQ is a focal chord of a parabola, focus S; two circles are drawn through S to touch the parabola at P, Q respectively; then they cut orthogonally.

159. P is any point on an ellipse; PSQ is a focal chord, PCP' is a diameter; prove that the pole of $P'Q$ lies on the auxiliary circle. [Use the property assumed in Th. 103, method (i.)]

160. D is any point on the circumcircle of an equilateral triangle ABC; prove that a parabola can be described having D as focus, and touching AB, BC, CA at their meets with DC, DA, DB respectively.

161. P, Q are two points on a focal chord of a parabola; PH, PK, QH, QK are the tangents from P, Q to the curve; prove that the angles PHQ, PKQ are equal or supplementary.

162. T is the pole of a chord PQ subtending a constant angle at the focus S of a conic; a line harmonically conjugate to PS w.r.t. PQ, PT meets ST at K; find the locus of K.

163. Two parabolas have a common focus and axis; a straight line is drawn through the focus; prove that the tangents at its meets with the parabolas form a rectangle, one diagonal of which passes through the focus, while the other is perpendicular to the axis.

164. On the tangent PT at any point P on a conic, a length PT is measured so as to subtend a right angle at a fixed point inside the conic; prove that the locus of T is the polar reciprocal w.r.t. O of the envelope of normals to the conic.

THEOREM 109.

(1) A system of conics, having a common focus, can be reciprocated into a system of circles; and if the latera recta of the conics are equal, the circles are of equal radii; and conversely.

(2) A system of conics, having a common focus and a common corresponding directrix, can be reciprocated into a system of concentric circles; and conversely.

The proof is left to the reader.

[Reciprocate w.r.t. the focus and use Theorems 107, 106.]

THEOREM 110.

(1) A system of confocal conics can be reciprocated into a system of coaxal circles.

(2) A system of coaxal circles, reciprocated w.r.t. a limiting point, becomes a system of confocal conics.

(1) Reciprocate w.r.t. one of the foci; then the conics become circles. But a system of confocals have, by definition, four fixed common tangents;

∴ the reciprocal system of circles have four common points.

[Two of these are the circular points and the other two lie on the radical axis.]

∴ the reciprocals are coaxal circles. Q.E.D.

(2) Let L, L' be the limiting points. Draw $L'H$ perpendicular to LL'.

Then $L'H$ is the polar of L w.r.t. each circle of the system.

Reciprocate w.r.t. L.

The circles become conics, having one focus at L.

But $L'H$, the polar of the origin w.r.t. each conic, reciprocates into the centre of each conic.

Therefore the reciprocals are conics having a common focus and centre, and are therefore confocal conics [p. 137, ex. 77]. Q.E.D.

Corollary.

The radical axis reciprocates into the second focus; and the second limiting point reciprocates into the minor axis.

[The radical axis is midway between the origin L and the line $L'H$, which becomes the centre.]

It is interesting to apply a more fundamental method of proof to Theorem 110.

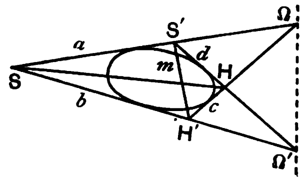

FIG. 100.

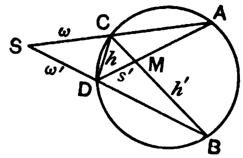

FIG. 101.

S, H; S', H' are the pairs of foci of a system of confocals; Ω, Ω' are the circular points at infinity.

Denote the lines $S\Omega$, $S\Omega'$, $H\Omega$, $H\Omega'$ by a, b, c, d.

Reciprocate w.r.t. S.

The reciprocal of $S\Omega$ (or a) is the point Ω, since the line $S\Omega$ touches any circle, centre S, at Ω. We denote this by A, in the ordinary way; similarly the reciprocal of $S\Omega'$ (or b) is Ω', which we denote by B.

Consequently in the reciprocal figure, A, B denote the circular points at infinity; and the lines ω, ω' are the reciprocals of Ω, Ω'; the lines c, d become points C, D on ω, ω'; the joins of C, D; C, B; A, D; are h, h', s'.

Any confocal, (since it touches a, b, c, d), becomes a conic through A, B, C, D, *i.e.* a circle through C, D.

Hence the reciprocal system is a set of coaxal circles having CD or h as radical axis. The radical axis is therefore the reciprocal of the second focus.

Moreover the point-circles belonging to this system are the isotropic pairs of lines ACS, BDS and AMD, BMC, which yield the point-circle S and the point-circle M; where M, the meet of AD, BC, is the reciprocal of m, the join of ad, bc or S', H', *i.e.* the minor axis. The limiting points of the coaxal system are therefore the focus S, taken as origin, and the reciprocal of the minor axis.

Since the figures are reciprocal in every respect, either may be regarded as generating the other, by reciprocation.

165. If two confocals intersect, prove that they cut orthogonally.

166. A system of conics have a common focus and a common corresponding directrix; prove that the polars of any given point w.r.t. the system are concurrent.

167. Tangents are drawn to a conic from two variable points on its directrix, which subtend a constant angle at the corresponding focus; prove that their meets lie on one of two conics, with the same focus and directrix.

168. A system of conics have a common focus, and touch each of two parallel lines: prove that the corresponding directrices are concurrent; that the centres are collinear; and that the asymptotes envelope a circle.

169. Reciprocate w.r.t. any point: if a variable circle touches two fixed circles, the polar of its centre w.r.t. one of the fixed circles envelopes a circle.

170. Reciprocate w.r.t. A: tangents are drawn from a fixed point A to a system of concentric circles, centre C; then the locus of the points of contact is a circle on AC as diameter.

171. What is the reciprocal of two homothetic conics, having a common focus, w.r.t. that focus?

172. A line drawn through a limiting point L of a coaxal system of circles cuts one of the circles at A, B. The tangents at A, B cut another circle of the system at P, Q and R, S respectively; prove that $P\hat{L}R = Q\hat{L}S$.

173. Two parabolas have a common focus and perpendicular axes; if the latus rectum of one is double that of the other, prove that triangles can be inscribed in one and also circumscribed to the other. [Assume the result of Th. 206.]

174. Reciprocate w.r.t. a focus: if tangents are drawn to a system of confocal conics from a fixed point on the major axis, the points of contact lie on a circle.

175. A system of hyperbolas have a common focus and a common corresponding directrix; find the envelope of the asymptotes.

176. Prove that confocal conics of reciprocal eccentricities intersect at the extremities of their latera recta.

177. Reciprocate w.r.t. S: a system of parabolas have a common focus S and a common tangent; then the points of contact of the other tangents from a fixed point on the common tangent, lie on a circle through S.

178. Reciprocate w.r.t. L: P, Q are conjugate points w.r.t. two circles of a given coaxal system; then they are conjugate points w.r.t. every circle of the system and if L is a limiting point, $P\hat{L}Q = 90°$.

179. (1) A system of similar conics have a common focus S; prove that their reciprocal w.r.t. S is a set of circles, such that the radius of any one is proportional to the distance of its centre from S.

(2) If a system of similar ellipses with a common focus S touch a fixed line, prove that the envelope of the corresponding directrix is another similar ellipse, with its centre on the given line and one focus at S.

180. Reciprocate w.r.t. any point: a variable conic has double contact with a fixed conic; if the chord of contact is fixed in direction, then the locus of its centre is a straight line.

181. Reciprocate w.r.t. any point: a variable circle passes through two fixed points, then the polar of a fixed point passes through another fixed point.

182. Prove that the locus of the pole of any tangent to the director circle of a conic w.r.t. that conic is another concentric conic.

183. Two conics have a common focus S, and a common corresponding directrix: a tangent at P to one meets the other at Q, R; prove that $Q\hat{S}P=P\hat{S}R$.

184. SL is the semi-latus rectum of a parabola, focus S; an ellipse is drawn through S to have four-point contact with the parabola at L; prove that it touches the axis of the parabola.

185. A variable circle passes through two fixed points A, B and cuts two fixed lines through A in P, Q; find the envelope of PQ. [Reciprocate w.r.t. B.]

186. One arm of a constant angle passes through a fixed point; the vertex moves on a fixed line; find the envelope of the other arm.

187. S is a focus of the conic σ_1; σ_2 is a conic having S as focus and any tangent of σ_1 as the corresponding directrix; if σ_2 touches the minor axis of σ_1, prove that the conics are of equal eccentricity.

188. The conics S_1, S_2 have a common focus and equal eccentricities: if the directrix of S_2 is an asymptote of S_1, prove that the minor axis of S_1 is an asymptote of S_2.

189. The tangents from a variable point P to two parabolas having a common focus and axis are at right angles; prove that the locus of P is a straight line.

THEOREM 111.

If S_1 and S_2 are two given conics, there exists in general a conic S, such that S_1, S_2 are reciprocal w.r.t. S.

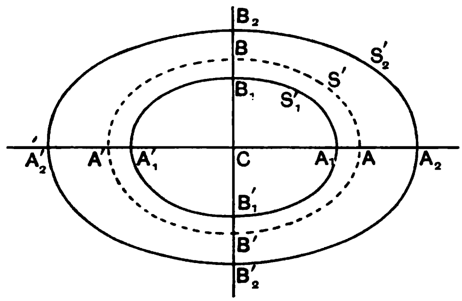

FIG. 102.

Project S_1 and S_2 into two conics S'_1, S'_2 having a common centre and the same principal axes. [If PQR is the common self-conjugate triangle, this is effected by projecting QR to infinity and $Q\hat{P}R$ into a right angle.]

Let A'_1CA_1, B'_1CB_1 and A'_2CA_2, B'_2CB_2 be the principal axes of S'_1, S'_2.

Take points A, A'; B, B' on CA_1, CB_1 such that

$$CA^2 = CA'^2 = CA_1 . CA_2 \text{ and } CB^2 = CB'^2 = CB_1 . CB_2.$$

Let S' be the conic having $A'CA$, $B'CB$ as principal axes.

Then the reciprocal of S'_1 w.r.t. S' passes through A_2, A'_2, B_2, B'_2 and has $A_2A'_2$, $B_2B'_2$ as principal axes, and therefore coincides with S'_2;

$\therefore$ S'_1 and S'_2 are reciprocal w.r.t. S'.

Therefore, projecting back, there exists a conic S w.r.t. which S_1 and S_2 are reciprocal. Q.E.D.

Corollary.

The three conics S_1, S_2, S have a common self-conjugate triangle.

190. If S_1, S_2 are two conics such that S_1 is its own reciprocal w.r.t. S_2, prove that S_1, S_2 have double contact with each other.

191. If a rectangular hyperbola is reciprocated w.r.t. a point O, prove that O lies on the director circle of the reciprocal conic.

What special case arises, if O lies on the rectangular hyperbola?

192. What is the reciprocal of a system of conics passing through four fixed points w.r.t. a vertex of the common self-conjugate triangle?

193. Reciprocate w.r.t. C: a variable circle is drawn through a fixed point and its radius is equal to that of a fixed circle, centre C; then the envelope of the common chord of the two circles is a conic, one of whose foci is at C.

194. The centre C of a circle S_1 is a vertex of a square circumscribing a circle S_2; prove that the reciprocal of S_2 w.r.t. S_1 is a rectangular hyperbola.

195. PQ is a variable chord of a rectangular hyperbola, subtending a right angle at a fixed point O, not on the curve; prove that PQ envelopes a parabola, having O as focus.

196. Reciprocate w.r.t. H: the orthocentre H of a triangle inscribed in a rectangular hyperbola lies on the curve.

197. Reciprocate w.r.t. O: if a conic touches the sides of a triangle, and passes through its circumcentre O, then its director circle touches the circumcircle.

198. A variable conic touches three fixed lines and its director circle passes through a fixed point; prove that the conic touches another fixed line. [Use Ex. 196.]

199. Prove that the reciprocal of a parabola w.r.t. a parabola is a conic with one asymptote parallel to the axis of the second parabola.

200. Reciprocate w.r.t. a focus: the envelope of the polars of a given point w.r.t. a system of confocal conics is a parabola touching the axes of the system.

201. Pairs of perpendicular chords are drawn through the vertex A of a conic: prove that the locus of the pole of the line through their other meets with the conic is a straight line, perpendicular to the axis. [Reciprocate w.r.t. A.]

202. If a fixed line meets a system of concentric circles, prove that the tangents at the points of intersection envelope a parabola.

203. Prove that the envelope of chords of an ellipse which subtend a right angle at the centre is a concentric circle.

204. Prove that a conic reciprocated w.r.t O becomes a rectangular hyperbola if, and only if, O lies on its director circle.

205. (1) Prove that any two conics can be reciprocated into rectangular hyperbolas. [Use Ex. 204.]

(2) Hence prove that the director circles of a system of conics touching four straight lines are coaxal.

206. Reciprocate w.r.t any point: P is a point on a rectangular hyperbola, centre C; the tangent at P meets an asymptote at T; then $P\hat{T}C = P\hat{C}T$.

207. p is a variable tangent to a fixed conic. P is the centre of the circle, which is the inverse of p w.r.t. a fixed point O; prove that the locus of P is a conic.

208. Prove that the reciprocal of a hyperbola, eccentricity e, w.r.t. a parabola having a common focus and directrix, is an ellipse of eccentricity $\frac{1}{e}$, with the same focus and directrix.

209. Two conics S_1, S_2 have the same asymptotes OA, OB; the polar of a variable point on S_1 w.r.t. S_2 meets OA, OB at P, Q; prove that the triangle OPQ is of constant area.

210. A hyperbola and a parabola have a common focus and touch one another; and their common chord, length $2c$, passes through the focus; if $2l$=latus rectum of parabola, prove that the eccentricity of the hyperbola is $\sqrt{5 \pm 4\sqrt{\frac{l}{c}}}$.

211. What is the reciprocal of a system of conics, touching four fixed lines, w.r.t. a vertex of the common self-conjugate triangle?

212. Reciprocate w.r.t. O: if a conic is inscribed in a fixed triangle and passes through a fixed point O, the locus of its centre is a conic touching the sides of the triangle formed by joining the mid-points of the sides of the fixed triangle.

213. Reciprocate w.r.t. O: O is a point inside a circle, centre A; the locus of the mid-points of chords of the circle, which subtend a right angle at O, is a circle, whose centre is at the mid-point of AO.

214. Reciprocate w.r.t. D: the centre of the rectangular hyperbola $ABCD$ lies on the auxiliary circle of the conic which touches the sides of the triangle ABC, and of which D is a focus.

215. With the focus of a hyperbola as centre, a circle is drawn, touching the asymptotes. P is the pole w.r.t. the hyperbola of a tangent to the circle; if this tangent meets the directrix at Q, prove that PQ touches the circle.

216. P is the pole of a chord QR of a conic S; prove that there are an unlimited number of conics, having PQR as a self-conjugate triangle, which are their own reciprocals w.r.t. S.

217. The incentre of a triangle self-conjugate to a hyperbola is at one of the foci; if e is the eccentricity, and l the semi-latus rectum, prove that the inradius$=\frac{l}{\sqrt{e^2-2}}$.

218. A system of conics have a common focus and corresponding directrix: prove that the normals to the conics at the extremities of the latera recta through the common focus touch a fixed parabola.

219. If the conic $S=L^2$ is reciprocated w.r.t. the conic $S=O$; $L=O$ being the equation of a given line; prove that, if the reciprocal of any point P on $S=L^2$ touches the reciprocal conic at P', then PP' passes through a fixed point.

220. Find the condition that $xy=c^2$ may be its own reciprocal w.r.t. $x^2+y^2=a^2$.

221. Each of the conics S_1, S_2 is its own reciprocal w.r.t. the other; prove that they have double contact with each other, and that they are also self-reciprocal to any conic having double contact with S_1 and S_2 at separate points of contact.

Analytical Treatment.

It is beyond the scope of this treatise to develop, or even to sketch the general theory of the inter-relation of point- and line-coordinates. But, by taking a special case of a simple algebraic character, it is possible to indicate quite briefly the manner in which the Principle of Duality presents itself in analysis.

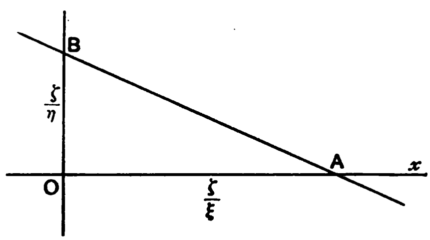

FIG. 103.

Ox, *Oy* are two rectangular axes. With the notation of page 24, let the coordinates of any point be $\left(\frac{x}{z}, \frac{y}{z}\right)$ where x, y, z are connected by the relation $x+y+z=1$.

Let any straight line cut *Ox*, *Oy* at A, B, and denote OA, OB by $\frac{\zeta}{\xi}, \frac{\zeta}{\eta}$ respectively, where $\xi+\eta+\zeta=1$.

The equation of AB is therefore $x\xi+y\eta=z\zeta$.

Now the line AB is known if ξ, η, ζ are given.

Consequently we shall call (ξ, η, ζ) the **coordinates of the line** AB.

If ξ, η, ζ vary, but always satisfy the equation

$$a\xi+b\eta=c\zeta,$$

where a, b, c are constants, we obtain a system of lines, having

ξ, η, ζ as coordinates; and each of these lines passes through the point, given by $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}=\dfrac{1}{a+b+c}$.

Consequently, we shall call $a\xi+b\eta=c\zeta$ the **equation of the point,** determined by this system of concurrent lines.

We now clearly have the analytical elements of a dual principle.

POINT-COORDINATES.	**LINE-COORDINATES.**
(f, g, h) is a point.	(f, g, h) is a line.
$fx+gy=hz$ is a line.	$f\xi+g\eta=h\zeta$ is a point.

Further the point (f, g, h) lies on the line $ax+by=cz$ if $af+bg=ch$: while the line (f, g, h) passes through the point $a\xi+b\eta=c\zeta$ if $af+bg=ch$: for the coordinates of all lines through the point $a\xi+b\eta=c\zeta$ must satisfy this equation, by the definition.

Similarly the points (x_1, y_1, z_1); (x_2, y_2, z_2); (x_3, y_3, z_3) are collinear if $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0$: while the lines (ξ_1, η_1, ζ_1); (ξ_2, η_2, ζ_2); (ξ_3, η_3, ζ_3) are concurrent if $\begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \\ \xi_3 & \eta_3 & \zeta_3 \end{vmatrix}=0$.

It is therefore clear that if any descriptive property is proved concerning the disposition of a system of points and lines in a plane, an exactly similar reciprocal theorem holds for the corresponding system of lines and points: for it is merely necessary to substitute coordinates of lines for coordinates of points, and equations of points for equations of lines in the previous analysis.

222. Find the point-equation of the line whose coordinates are $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$.

223. Find the point-coordinates of a point whose line-equation is (1) $2\xi-3\eta+4\zeta=0$; (2) $5\xi=2\zeta$.

224. Find the line-coordinates of the lines whose point-equations are (1) $y=mx+cz$; (2) $ax+by=0$; (3) $x=0$; (4) $x\cos\alpha+y\sin\alpha=pz$; (5) $\dfrac{x}{a}+\dfrac{y}{b}=\dfrac{z}{c}$; (6) the line at infinity, $z=0$.

225. Find the line-equations of the points whose point-coordinates are (1) $0, 1, 0$; (2) $2, 0, -1$; (3) the origin; (4) the point at infinity on $y=2x$; (5) the point at infinity on $y=z$.

226. Find the point-coordinates of the fixed point on the variable line whose line-coordinates are $\xi=2+3t$, $\eta=3-2t$, where t varies.

227. Find the line-coordinates of the line traced out by the moving point, whose point-coordinates are $(1+t, 1-2t)$, where t varies.

228. Find the line-coordinates of the line joining the points, whose point-coordinates are $(2, 3)$; $(-4, 5)$.

229. Find the line-equation of the meet of the lines, whose point-equations are $2x+3y=z$, $3x+2y=7z$.

230. Prove that the line-equations $a\xi+b\eta+c\zeta=0$, $a\xi+b\eta+d\zeta=0$ represent points collinear with the origin.

231. Prove that the lines whose coordinates are (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are parallel if $\dfrac{\xi_1}{\xi_2}=\dfrac{\eta_1}{\eta_2}$.

232. Given the line-coordinates of three lines, determine the condition that the lines are concurrent.

233. Given the line-equations of three points, determine the condition that the points are collinear.

The introduction of z and ζ has been made, in order to secure a complete representation of elements, finite and ideal, in the x, y plane. Thus the line at infinity is represented by the equation $z=0$; and the line-equation of the origin is $\zeta=0$. It often happens however that this wider scope of interpretation is of little value to the problem in hand, and merely serves to complicate the analysis. When therefore it is more convenient, z and ζ will be omitted; and in that case x, y, ξ, η will stand for $\dfrac{x}{z}$, $\dfrac{y}{z}$, $\dfrac{\xi}{\zeta}$, $\dfrac{\eta}{\zeta}$, respectively.

234. (1) Any point on the join of the points (x_1, y_1), (x_2, y_2) can be represented by $\left(\dfrac{x_1+\lambda x_2}{1+\lambda}, \dfrac{y_1+\lambda y_2}{1+\lambda}\right)$, for a suitable value of λ.

(2) Any line through the meet of the lines (ξ_1, η_1), (ξ_2, η_2) can be represented by $\left(\dfrac{\xi_1+\lambda\xi_2}{1+\lambda}, \dfrac{\eta_1+\lambda\eta_2}{1+\lambda}\right)$, for a suitable value of λ.

235. (1) The cross ratio of the four points (x_1, y_1), $\left(\dfrac{x_1+\lambda x_2}{1+\lambda}, \dfrac{y_1+\lambda y_2}{1+\lambda}\right)$, (x_2, y_2), $\left(\dfrac{x_1+\mu x_2}{1+\mu}, \dfrac{y_1+\mu y_2}{1+\mu}\right)$ is $\dfrac{\lambda}{\mu}$.

(2) The cross ratio of the four lines (ξ_1, η_1), $\left(\dfrac{\xi_1+\lambda\xi_2}{1+\lambda}, \dfrac{\eta_1+\lambda\eta_2}{1+\lambda}\right)$, (ξ_2, η_2), $\left(\dfrac{\xi_1+\mu\xi_2}{1+\mu}, \dfrac{\eta_1+\mu\eta_2}{1+\mu}\right)$ is $\dfrac{\lambda}{\mu}$.

POINT RECIPROCATION.

Consider now the base-circle $x^2+y^2=1$.

Let (ξ', η') be the line-coordinates of the polar of any point (x', y').

Then (ξ', η') are the coordinates of the line $xx'+yy'=1$.

$$\therefore\ \xi'=x',\ \eta'=y'.$$

Suppose that (x', y') traces out the curve $f(x, y)=0$.

Then the coordinates (ξ, η) of its polar satisfy the equation $f(\xi, \eta)=0$.

Now this line (ξ, η) envelopes the reciprocal of the curve $f(x, y)=0$ w.r.t. the circle $x^2+y^2=1$.

Consequently $f(\xi, \eta)=0$ is the **line-equation of the reciprocal** of $f(x, y)=0$ w.r.t. $x^2+y^2=1$.

The following example illustrates the method for deducing the point-equation from a given line-equation.

EXAMPLE.

To find the point-equation of the reciprocal of the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ w.r.t. the circle $x^2+y^2=1$.

The line-equation of the reciprocal is

$$a\xi^2+2h\xi\eta+b\eta^2+2g\xi+2f\eta+c=0.$$

Let (ξ, η) be the coordinates of any line through the point (x', y').

$$\text{Then } \xi x'+\eta y'=1. \quad\ldots\ldots\ldots\ldots(1)$$

If this line (ξ, η) also touches the reciprocal, we have

$$a\xi^2+2h\xi\eta+b\eta^2+2g\xi+2f\eta+c=0. \quad\ldots\ldots\ldots\ldots(2)$$

Eliminating η from (1) and (2) we have

$$\xi^2(ay'^2-2hx'y'+bx'^2)+2\xi(gy'^2-fx'y'+hy'-bx')+cy'^2+2fy'+b=0.$$

Now if (x', y') lies on the reciprocal curve, the two tangents that can be drawn from it to the curve are coincident: and therefore, in this case, the quadratic in ξ must have equal roots.

$$\therefore\ (ay'^2-2hx'y'+bx'^2)(cy'^2+2fy'+b)=(gy'^2-fx'y'+hy'-bx')^2.$$

Simplifying, and dividing through by y'^2, we obtain as the locus of (x', y'):

$$x^2(bc-f^2)+2xy(fg-ch)+y^2(ac-g^2)-2x(fh-bg)$$
$$-2y(gh-af)+ab-h^2=0;$$

or, with the usual notation,

$$Ax^2+2Hxy+By^2-2Gx-2Fy+C=0,$$

where A, B, C, F, G, H, are the cofactors of a, b, c, f, g, h, in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Q.E.F.

If $S_1(x, y)=0$, $S_2(x, y)=0$, denote the point-equations of two conics, we know that any given conic through their four common points is represented by an equation of the form $S_1+\lambda S_2=0$, where λ is a suitably chosen constant.

In precisely the same way, if $\Sigma_1(\xi, \eta)=0$, $\Sigma_2(\xi, \eta)=0$ denote the line-equations of two conics, it follows that any given conic touching their four common tangents is represented by an equation of the form $\Sigma_1+\lambda\Sigma_2=0$, for a suitable value of the constant λ.

Now the coordinates of the isotropic lines $y=\pm ix+d$ are given by $\frac{\xi}{\pm i}=\frac{\eta}{-1}=\frac{\zeta}{d}$, so that the coordinates of all isotropic lines are subject to the relation $\xi^2+\eta^2=0$, or in other words, the equation of the pair of circular points at infinity is $\xi^2+\eta^2=0$.

But confocal conics are, by definition, a system of conics touching two fixed pairs of isotropic lines.

Consequently, if $\Sigma(\xi, \eta)=0$ is the line-equation of any conic, then any given conic confocal with it is represented by an equation of the form $\Sigma(\xi, \eta)+\lambda(\xi^2+\eta^2)=0$, for a suitable value of the constant λ.

It is interesting to verify analytically the theorem that the reciprocal of a system of coaxal circles w.r.t. a limiting point is a system of confocal conics.

Take the origin at one limiting point, and let $(a, 0)$ be the coordinates of the other.

Then $x^2+y^2=0$ and $(x-a)^2+y^2=0$ are two circles of the coaxal system.

Therefore any other circle of the system is represented by

$$\lambda(x^2+y^2)+(x-a)^2+y^2=0.$$

Therefore the line equation of the reciprocal w.r.t. $x^2+y^2=1$ is

$$\lambda(\xi^2+\eta^2)+(\xi-a)^2+\eta^2=0.$$

which represents one of a system of confocal conics, by our previous work.

Q.E.D.

The reader should note that this is simply the analytical statement of the mode of proof adopted on pages 192–3.

236. Find the line-coordinates of the common tangents of the circles $x^2+y^2=1$; $x^2-2x+y^2=0$.

237. Find the line-coordinates of the common tangents of the conics $2x^2+3y^2=1$; $3x^2+2y^2=1$.

238. Find the line-equation of the reciprocal of the conic $2x^2-3y^2=1$ w.r.t. the circle $x^2+y^2=1$; and deduce its point-equation.

239. Find the line-coordinates of the tangents from the point (2, 3) to the conic whose line-equation is $\xi^2+2\xi\eta+3\eta^2=1$.

240. Find the line-equation of the reciprocal of $x^2+3xy+2y^2-2x=4$ w.r.t. the conic $x^2+3y^2=1$; and deduce its point-equation.

241. Write down the general line-equation of a parabola. [Note that it touches the line at infinity ; $\xi=0$, $\eta=0$.]

242. If $\Sigma=0$ is the line-equation of a conic and if $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ are the line-equations of four points, interpret geometrically the equations (1) $\Sigma-k.\alpha\beta=0$; (2) $\Sigma-k\alpha^2=0$; (3) $\Sigma-k\alpha=0$; (4) $\Sigma-k=0$; (5) $\alpha\beta-k.\gamma\delta=0$; where k is a constant.

243. Calculate the line-equation of the system of confocal conics given by $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$, where λ varies.

244. Prove that the general point-equation of a conic confocal with $ax^2+2hxy+by^2=1$ is $(a+\lambda)x^2+2hxy+(b+\lambda)y^2+\dfrac{h^2-(a+\lambda)(b+\lambda)}{ab-h^2}=0$.

[Calculate the line-equation of the given conic ; deduce the general equation of a confocal ; and then turn it back into point-coordinates.]

245. (1) Find the line-equation of the pole of ξ, η w.r.t. the conic whose line-equation is $a\xi^2+2h\xi\eta+b\eta^2+2g\xi+2f\eta+c=0$.

(2) Hence find the equation of the centre of the conic.

(3) Taking the general equation of a conic touching four lines in the line-form $\alpha\beta-k.\gamma\delta=0$, prove that the locus of the centres of conics touching four fixed lines is a straight line.

246. Work out an analytical proof of Brianchon's theorem, in line-coordinates, similar to that of Pascal's theorem in point-coordinates on page 150.

CHAPTER VIII.

HOMOGRAPHIC RANGES AND PENCILS.

It has already been pointed out that a knowledge of the fundamental property of the cross ratio of a pencil of four concurrent lines dates back to Pappus, but that the general theory of ranges and pencils is essentially modern. It is difficult to estimate the precise contribution of Desargues and his followers, few in number but of great ability. As the method of projection came into more general use, greater attention was naturally directed towards the theory of cross ratios. And these researches were crowned by the comprehensive *Géometrie Supérieure* of Chasles, published in 1852. This treatise, of remarkable originality, brings to the theory of homography a generality of statement and a sense of power, which place it among the greatest books of the last century. His discovery of the double points of cobasal homographic ranges leads him to an ingenious method of solving a wide group of constructions, and the general theory of involution, which is seen to be a special case of homography, provides a new means of introducing imaginary elements into pure geometry. To weigh however the influence of Chasles on the progress of geometrical research, it is necessary to take into account his invaluable historical investigations of the work done in former centuries, contained in his *Aperçu Historique*, which doubtless stimulated his own thought, as well as that of his contemporaries.

For convenience of reference, we shall first enumerate certain cross ratio properties which have been previously enunciated. [See Part I. pages 52, 103-107.]

(1) The cross ratio of the range formed by four collinear points A, B, C, D is $\{ABCD\} \equiv \dfrac{AB \cdot CD}{AD \cdot CB}$.

The cross ratio of the pencil formed by four concurrent lines a, b, c, d is $\{abcd\} \equiv \dfrac{\sin \widehat{ab} \cdot \sin \widehat{cd}}{\sin \widehat{ad} \cdot \sin \widehat{cb}}$.

[$\widehat{ab}$ denotes the angle b makes with a.]

(2) If the joins of corresponding points of two 4-point ranges are concurrent, the ranges are equicross.

If the meets of corresponding rays of two 4-ray pencils are collinear, the pencils are equicross.

(3) If two equicross 4-point ranges, on different bases, have a self-corresponding point, the joins of the other corresponding points are concurrent.

If two equicross 4-ray pencils, with different vertices, have a self-corresponding ray, the meets of the other corresponding rays are collinear.

(4) $\{ABCD\}$ is unaltered in value, if, when any two letters are interchanged, the other two letters are also interchanged.

$\{abcd\}$ is unaltered in value, if, when any two letters are interchanged, the other two letters are also interchanged.

(5) If $\{ACBD\}=\{ADBC\}$, the range $\{AB;\ CD\}$ is harmonic.

If $\{acbd\}=\{adbc\}$, the pencil $\{ab;\ cd\}$ is harmonic.

(6) If A, B, C are three fixed collinear points, and if k is a constant, then there is *one and only one* point X such that $\{ABCX\}=k$.

If a, b, c are three fixed concurrent lines, and if k is a constant, then there is *one and only one* line x such that $\{abcx\}=k$.

ANALYTICAL TREATMENT OF HOMOGRAPHIC RANGES.

(I.) A, B, C are three fixed points on a base l; A', B', C' are three fixed points on a base l'; O, O' are fixed points, taken as origins on l, l'.

X is a variable point on l; X' is the point on l' for which $\{ABCX\}=\{A'B'C'X'\}$.

It is required to find the relation connecting the positions of X, X'.

Let $OA=a$, $OX=x$; $O'A'=a'$, $O'X'=x'$; etc.

By hypothesis, $\dfrac{AB\,.\,CX}{AX\,.\,CB}=\dfrac{A'B'\,.\,C'X'}{A'X'\,.\,C'B'}$; but $CX=x-c$, etc.;

$$\therefore\ k\frac{x-c}{x-a}=k'\frac{x'-c'}{x'-a'},\ \text{where } k,\ k' \text{ are constants.}$$

$$\therefore\ pxx'+qx+rx'+s=0,$$

where p, q, r, s are constants; which is the required relation.

Q.E.F.

(II.) O, O' are fixed origins on the fixed lines l, l'; X, X' are variable points on l, l', subject to the condition

$$pxx'+qx+rx'+s=0,$$

where $OX=x$, $OX'=x'$; p, q, r, s being constants. Then the range formed by any four positions of X is equicross with the range formed by the four corresponding positions of X'.

Let x_1, x_2, x_3, x_4 be the coordinates of any four points on l, and let x'_1, x'_2, x'_3, x'_4 be the corresponding points on l'.

Now $$\{x_1x_2x_3x_4\} = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_1)(x_2 - x_3)}.$$

But $$x_1 = -\frac{rx'_1 + s}{px'_1 + q}, \text{ by definition.}$$

Therefore

$$x_2 - x_1 = \frac{rx'_1 + s}{px'_1 + q} - \frac{rx'_2 + s}{px'_2 + q} = \frac{(rx'_1 + s)(px'_2 + q) - (rx'_2 + s)(px'_1 + q)}{(px'_1 + q)(px'_2 + q)}$$
$$= \frac{(x'_2 - x'_1)(ps - qr)}{(px'_1 + q)(px'_2 + q)}.$$

Similarly $$x_4 - x_3 = \frac{(x'_4 - x'_3)(ps - qr)}{(px'_3 + q)(px'_4 + q)}, \text{ etc.}$$

Therefore, after simplifying,

$$\{x_1x_2x_3x_4\} = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_1)(x_2 - x_3)} = \frac{(x'_2 - x'_1)(x'_4 - x'_3)}{(x'_4 - x'_1)(x'_2 - x'_3)}$$
$$= \{x'_1x'_2x'_3x'_4\}. \qquad \text{Q.E.D.}$$

We may prove (II.) in another way:

With the same notation as before, let ξ, ξ' be any pair of points such that $\{x_1x_2x_3\xi\} = \{x'_1x'_2x'_3\xi'\}$.

This relation is identically satisfied by

$$\left.\begin{matrix}\xi = x_1 \\ \xi' = x'_1\end{matrix}\right\}; \quad \left.\begin{matrix}\xi = x_2 \\ \xi' = x'_2\end{matrix}\right\}; \quad \left.\begin{matrix}\xi = x_3 \\ \xi' = x'_3\end{matrix}\right\}$$

Now by (I.), the relation reduces to the form

$$P\xi\xi' + Q\xi + R\xi' + S = 0.$$

The values $P:Q:R:S$ are then determined *uniquely*, in terms of x_1, x_2, x_3, x'_1, x'_2, x'_3, by substituting for ξ, ξ' their three pairs of values.

But the equation $pxx' + qx + rx' + s = 0$ is satisfied by the same three pairs of values $\left.\begin{matrix}x = x_1 \\ x = x'_1\end{matrix}\right\}$, etc.

$$\therefore \; P:Q:R:S = p:q:r:s.$$

Therefore the value of ξ' arising from $\{x'_1x'_2x'_3\xi'\} = \{x_1x_2x_3x_4\}$ is the same as the value of x' arising from $px_4x' + qx_4 + rx' + s = 0$ and is therefore equal to x'_4;

$$\therefore \; \{x_1x_2x_3x_4\} = \{x'_1x'_2x'_3x'_4\}. \qquad \text{Q.E.D.}$$

Definition.

If $A, B, C, \ldots X, \ldots$; $A', B', C', \ldots X', \ldots$, are two ranges of points on the same or different bases: and if the cross ratio of any four

points P, Q, R, S of one range is equal to that of the four corresponding points P', Q', R', S' of the other, the two ranges are said to be **homographic**.

In (I.), it is proved that any pair of corresponding points X, X' of two given homographic ranges are connected by a relation of the form $pxx'+qx+rx'+s=0$, where p, q, r, s are constants, and x, x' denote the distances of X, X' from fixed origins on the bases. Conversely, in (II.), it is proved that, if two ranges are generated by a pair of points X, X' moving so that their distances $OX=x$, $O'X'=x'$, from fixed origins O, O' on the two bases are connected by a relation of the form $pxx'+qx+rx'+s=0$, where p, q, r, s are constants, then the two ranges are homographic.

These two results may be stated in one theorem:

The existence of an equation of the form $pxx'+qx+rx'+s=0$ is the necessary and sufficient condition that two ranges, generated from it, are homographic.

The fundamental characteristic of the homographic relation is the fact that it sets up a one-to-one correspondence. To any point of either range, there corresponds one and only one point of the other range. This is put in evidence very clearly by the relation, which is linear in each of the variables x, x'; but it is embodied, with equal precision, in the cross-ratio property on which the definition is based.

1. $OABC$, $OA'B'C'$ are two lines, such that $OA=1$, $OB=2$, $OC=3$ and $OA'=1$, $OB'=3$, $OC'=6$. Prove that the equation determining the homographic ranges $\{ABC\ldots\}, \{A',B',C',\ldots\}$ is $xx'+9x-7x'-3=0$.

2. In Ex. 1, if the two lines coincide, with OA on OA', determine another point besides A on the line, which corresponds to itself in the two ranges.

3. Two homographic ranges are defined by the relation

$$xx'+x-x'+1=0;$$

determine the points in the second range corresponding to

$$x=-1,\ +1,\ 0,\ \infty \text{ and } x=-2,\ +2,\ -3,\ +3.$$

Verify that the cross ratio of each set of four points is the same as the cross ratio of the corresponding points.

4. With the relation of Ex. 3, if I and J' are the points on l, l' which correspond to the points at infinity on l', l respectively, and if X, X' are a variable pair of corresponding points, prove that $XI \cdot X'J'$ is constant.

5. Determine the relation defining two homographic ranges on the same base, such that the points 2, -5 correspond to themselves in the two ranges and the point -2 in the first corresponds to the point 1 in the second.

6. Two homographic ranges, on the same base, with the same origin, are defined by the relation $xx'-x-4x'+6=0$. Find the points of the second range corresponding to the points $0, \pm 1, \pm 2, \infty$ of the first, and find the points of the first range corresponding to the points $0, \pm 1, \pm 2, \infty$ of the second.

Prove that there are two points α, β on the base which are self-corresponding for the two ranges.

7. Two homographic ranges, on the same base, with the same origin, are defined by the relation $xx'-x-4x'+6=0$; a is any point on the base at distance ξ from the origin; A', A are the two points which correspond to a according as it is regarded as a point of the x-range or the x'-range; prove that $A'A=\dfrac{3(\xi-2)(\xi-3)}{(\xi-1)(\xi-4)}$.

8. With the notation of Ex. 7, if the homographic relation is $pxx'+qx+rx'+s=0$, prove that $A'A=\dfrac{(q-r)[p\xi^2+(q+r)\xi+s]}{(p\xi+q)(p\xi+r)}$.

Hence prove that the necessary and sufficient condition that, if X, X' are any pair of corresponding points in the first and second range, then also X', X are corresponding points in the first and second range, is $q=r$.

Deduce this also from first principles.

9. With the notation of Ex. 6, if I and J' are the points corresponding to the point at infinity regarded as belonging to the x'-range and x-range respectively, prove that αI, $\beta J'$ are of equal length.

10. Two homographic ranges $\{ABC\ldots\}$, $\{A'B'C'\ldots\}$ are connected by the relation $qx+rx'+s=0$; prove that $\dfrac{AB}{A'B'}=\dfrac{BC}{B'C'}=\dfrac{CD}{C'D'}\cdots$

11. OAB is a straight line; $OA=a$, $OB=b$; determine the relation defining the homographic ranges $\{OA\infty\ \ldots,\ \{\infty BO\ldots\}$. If X, X'; Y, Y' are any two pairs of corresponding points, prove that

$$\{AXYY'\}=\{BX'Y'Y\}.$$

12. Prove that the result of Ex. 9 holds good for any two cobasal homographic ranges.

13. If the x-range and the x'-range are connected by the relation $pxx'+qx+rx'+s=0$, and if the x-range and the x''-range are connected by the relation $Pxx''+Qx+Rx''+S=0$, find the relation between the x'-range and the x''-range.

If two ranges are homographic to the same range, prove that they are homographic to each other.

14. With the notation of Ex. 13, if the three ranges are cobasal and if the same two points are self-corresponding in the two pairs of ranges, prove that $\dfrac{p}{P}=\dfrac{q+r}{Q+R}=\dfrac{s}{S}$.

15. Two homographic ranges are determined by the relation,

$$pxx'+qx+rx'+s=0;$$

prove that the points at infinity in the two ranges correspond to each other, if and only if $p=0$.

16. α, β are the two self-corresponding points of the two ranges on the same base, referred to the same origin, determined by

$$pxx'+qx+rx'+s=0;$$

if X, X' are a variable pair of corresponding points, prove that $\{\alpha X\beta X'\}$ is constant.

17. With the notation of Ex. 16, find the condition that $\{\alpha\beta\,;\,XX'\}$ is harmonic.

18. Two homographic ranges, whose bases are the x-axis and y-axis, are defined by the relation $xy-x-4y+6=0$. Find the points P, Q' corresponding to the origin for the two ranges.

If R, R'; S, S' are any other two pairs of corresponding points, prove that RS', $R'S$, meet on PQ'.

If I, J' correspond to the points at infinity on the two bases, prove that PQ' is parallel to IJ'.

19. With the notation of Ex. 18, if the homographic relation is $xy-px-qy+r=0$, prove that RS', $R'S$, meet on the fixed line $px+qy=r$.

20. P, P' are a pair of corresponding points of the ranges determined by $pxx'+q(x+x')+s=0$; if the coordinates of P, P' are the roots of the equation $ax^2+2bx+c=0$, prove that $pc+sa=2qb$.

If E, F are the self-corresponding points of the two ranges, prove that $\{PP'\,;\,EF\}$ is harmonic.

21. Two cobasal homographic ranges

$$\{ABCI\infty\, D\ldots\},\ \{A'B'C'\infty J'D'\ldots\},$$

are defined by $xx'+qx+rx'+s=0$. If the origin is transferred to the mid-point O of IJ', prove that the relation becomes $xx'-q'(x-x')+s'=0$, where $IO=OJ'=q'$ and $s'=-IO\,.\,OO'$, O' being the point of the x'-range which corresponds to O in the x-range.

If E, F are the self-corresponding points, deduce that

$$EO^2=OF^2=IO\,.\,OO'=OJ'\,.\,OO'.$$

DOUBLE POINTS.

Consider two homographic ranges on the same base, defined by the relation $pxx'+qx+rx'+s=0$, referred to the same origin.

A pair of corresponding points coincide if $x=x'$; in which case,

$$px^2+(q+r)x+s=0.$$

This equation gives two values for x, which are real, coincident, or imaginary, according as $(q+r)^2-4ps>,\ =,\ <0$.

Definition.

If a point on the common base of two homographic ranges corresponds to itself for the two ranges, it is called a **double point.**

(III.) Two homographic ranges on the same base have always two double points which may be real, coincident or imaginary. And if there are more than two double points, then every point is a double point.

Since the equation $px^2+(q+r)x+s=0$ is of the second degree, there are always two double points. If, however, there are more than two double points, we have a quadratic satisfied by more than two values of x, and therefore each coefficient must be zero;

$$\therefore \; p=0; \; q+r=0; \; s=0,$$

so that the homographic relation becomes

$$q(x-x')=0 \text{ or } x=x',$$

which means that every point is a double point, a case of small interest. Q.E.D.

(IV.) If E, F are the double points, and if A, A' are a variable pair of corresponding points, then $\{EAFA'\}$ is constant.

Let B, B' be any other pair of corresponding points.

With the usual notation (p. 206), from II., we have

$$\{EAFB\}=\{EA'FB'\};$$

$$\therefore \; \frac{(a-e)(b-f)}{(b-e)(a-f)}=\frac{(a'-e)(b'-f)}{(b'-e)(a'-f)};$$

$$\therefore \; \frac{(a-e)(a'-f)}{(a'-e)(a-f)}=\frac{(b-e)(b'-f)}{(b'-e)(b-f)};$$

$$\therefore \; \{EAFA'\}=\{EBFB'\};$$

$$\therefore \; \{EAFA'\} \text{ is constant.}$$ Q.E.D.

22. With the usual notation, prove that $\{EAFA'\}=\dfrac{q+pf}{q+pe}=\dfrac{r+pe}{r+pf}$.

23. With the usual notation, if $\{EAFA'\}$ is harmonic, prove that $q=r$. State and prove the converse.

(V.) If one of the double points is at infinity, the ranges are **similar**, *i.e.* the line joining any two points is proportional to the line joining the corresponding points.

By hypothesis $x \to \infty$ must satisfy the equation

$$px^2+(q+r)x+s=0;$$

$\therefore$ $\xi \to 0$ must satisfy the equation $p+(q+r)\xi+s\xi^2=0$.

This requires that $p=0$.

If y, y'; z, z' denote two pairs of corresponding points,

$$qy + ry' + s = 0 \text{ and } qz + rz' + s = 0;$$

$$\therefore \; q(y-z) + r(y'-z') = 0;$$

$$\therefore \; \frac{y-z}{y'-z'} = -\frac{r}{q} = \text{constant.} \qquad \text{Q.E.D.}$$

(VI.) If two ranges on the same base are similar, the point at infinity is a double point.

Let A, A' be a fixed pair and X, X' a variable pair of corresponding points.

Then, with the usual notation, $\dfrac{x-a}{x'-a'}$ is constant;

$$\therefore \; x - rx' = s, \text{ where } r, s \text{ are constants;}$$

$\therefore$ one double point is at infinity. Q.E.D.

24. Two similar ranges are defined by the relation $qx+rx'=s$; find the condition that the second double point is (1) at infinity, (2) at the origin. What is the geometrical interpretation of the former case?

25. Two similar ranges, situated on the lines OA, OA', are defined by the relation $qx+rx'=s'$, referred to the common origin O. If P, P' are a variable pair of corresponding points, find the locus of the mid-point of PP'.

26. Prove that a variable tangent to a parabola generates similar ranges on two fixed tangents. [With the fixed tangents as axes, the equation of the parabola may be written $\sqrt{ax}+\sqrt{by}=1$.]

ANALYTICAL TREATMENT OF HOMOGRAPHIC PENCILS.

The properties of homographic pencils may be treated in a similar manner.

Let $y=m_1x$, $y=m_2x$, $y=m_3x$ and $Y=M_1X$, $Y=M_2X$, $Y=M_3X$ be two sets of three concurrent lines referred to the same or different axes. To a variable line $y=m_kx$ of the first set, we make a line $Y=M_kX$ of the second set correspond, choosing it so that the cross ratios of the two sets of lines are equal.

M_k is therefore given by

$$\frac{(M_2-M_1)(M_k-M_3)}{(M_k-M_1)(M_2-M_3)} = \frac{(m_2-m_1)(m_k-m_3)}{(m_k-m_1)(m_2-m_3)}.$$

This leads to a relation of the form $pM_k . m_k + qM_k + rm_k + s = 0$, where p, q, r, s are constants.

This relation has the characteristic property, that to any value of m_k, there corresponds one and only one value of M_k, and conversely. The correspondence is therefore one-to-one, (1, 1) or homographic.

Conversely, if any two pencils of concurrent lines are connected by a relation of the form

$$pM\,.\,m+qM+rm+s=0,$$

the cross ratio of any four lines of one pencil is equal to that of the corresponding four lines of the other.

This is proved in precisely the same way as the similar theorem for ranges (page 207).

It should be noted that the two pencils intercept on the lines $x=1$, $X=1$, ranges whose corresponding points are determined by $y=m$, $Y=M$. Hence any theorem relating to ranges can be transformed into a property of pencils.

Definition.

If two systems of concurrent lines are so related that the cross ratio of any four rays of one system is equal to the cross ratio of the corresponding rays of the other, then the two systems are said to form **homographic pencils.**

From our previous work we have the following important theorem:

The existence of an equation of the form pMm + qM + rm + s = 0 is the necessary and sufficient condition that two pencils, generated from it, are homographic.

27. Determine the homographic relation, for which $y=x$, $y=2x$, $y=3x$ and $Y=X$, $Y=3X$, $Y=6X$ are sets of corresponding lines; and find the lines corresponding to $y=0$ and $x=0$.

If the axes of reference coincide, find the two rays which are self-corresponding for the two pencils.

28. The relation defining two homographic pencils is

$$Mm+M-m+1=0;$$

find the four rays corresponding to $X=0$, $Y=0$, $Y=X$, $Y=-X$; and prove that they form a harmonic pencil.

29. $pMm+qM+rm+s=0$ is the homographic relation, connecting two pencils, with the same vertex and axes; $y=Hx$, $y=kx$ correspond to any line $y=\alpha x$, regarded as belonging to the m-system and M-system respectively; prove that $H-k=\dfrac{(q-r)[p\alpha^2+(q+r)\alpha+s]}{(p\alpha+q)(p\alpha+r)}$.

Deduce a theorem for pencils defined by the relation

$$pMm+q(M+m)+s=0.$$

30. If each of two pencils is homographic to a third pencil, prove that they are homographic to each other.

31. OE, OF are the two self-corresponding rays of two pencils, defined by $pMm+qM+rm+s=0$, referred to the same axes; OP, OP' are any pair of corresponding rays; prove that $O\{EPFP'\}$ is constant.

DOUBLE RAYS.

Consider two homographic pencils, with the same vertex, defined by the relation $pMm+qM+rm+s=0$, referred to the same axes.

A pair of corresponding rays coincide if $M=m$, in which case,

$$pm^2+(q+r)m+s=0.$$

This equation gives two values for m, which are real, coincident, or imaginary, according as $(q+r)^2-4ps>, \ =, \ <0$.

Definition.

If a ray through the common vertex of two homographic pencils corresponds to itself for the two pencils, it is called a **double ray.**

(VII.) Two homographic pencils with the same vertex have always two double rays, which may be real, coincident, or imaginary. And if there are more than two double rays, then every ray is a double ray.

The proof is left to the reader (see page 211).

(VIII.) If e, f are the double rays, and if a, a' are a variable pair of corresponding rays, then $\{eafa'\}$ is constant.

The proof is left to the reader (see page 211).

(IX.) An angle POP' of constant magnitude rotates about a fixed vertex O; then the rays OP, OP' generate homographic pencils, having as double rays the isotropic lines through O.

Let $y=Mx$, $y=mx$ be any two positions of OP, OP'.

By hypothesis, $\dfrac{M-m}{1+Mm}=\text{constant}=c$, (say);

$$\therefore \quad cMm-M+m+c=0;$$

$\therefore$ the rays generate homographic pencils.

The double rays are found by putting $M=m=\mu$;

$$\therefore \quad c\mu^2+c=0; \qquad \therefore \quad \mu^2=-1 \text{ since } c\neq 0.$$

[If $c=0$, the lines OP, OP' would coincide.]

$$\therefore \quad \mu=\pm\sqrt{-1}\equiv\pm i;$$

$\therefore$ $y=ix$; $y=-ix$ are the double rays. Q.E.D.

(X.) If the isotropic lines are the double rays of two homographic pencils with the same vertex, then corresponding rays are inclined at a constant angle.

Let the homographic relation be $pMm+qM+rm+s=0$.

Then the double rays are given by $p\mu^2+(q+r)\mu+s=0$.

But by hypothesis, this is the same as $\mu^2+1=0$;

$$\therefore \quad p=s \text{ and } q+r=0;$$

$\therefore$ the homographic relation becomes $p(Mm+1)=r(M-m)$;

$$\therefore \frac{M-m}{1+Mm}=\frac{p}{r}=\text{constant}. \qquad \text{Q.E.D.}$$

(XI.) (1) If $P\hat{O}P'$ is a variable angle of constant magnitude, and if OI, OJ are the isotropic lines through O, the pencil $O\{PIP'J\}$ is of constant cross ratio.

(2) Conversely, if the pencil $O\{PIP'J\}$ is of constant cross ratio, where OI, OJ are the isotropic lines through O, the angle $P\hat{O}P$ is of constant magnitude.

In either case OI, OJ are the double rays of the homographic pencils generated by OP, OP'. The first part therefore follows from (VIII.) and the second part from (X.).

The reader will find no difficulty in giving a direct analytical proof, see Ex. 32, 33.

32. Prove that the cross ratio of the lines $y=mx$, $y=ix$, $y=m'x$, $y=-ix$ is constant, if $\dfrac{m-m'}{1+mm'}$ is constant; and prove that they form a harmonic pencil if $y=mx$, $y=m'x$ are at right angles.

33. If the cross ratio of the lines $y=mx$, $y=ix$, $y=m'x$, $y=-ix$ is constant, prove that $y=mx$, $y=m'x$ are inclined to each other at a constant angle.

34. If the cross ratio of the lines $y=mx$, $y=ix$, $y=m'x$, $y=-ix$ is k, prove that the angle between the lines $y=mx$, $y=m'x$ is $\dfrac{1}{2i}\log k$ radians. [This result is due to Laguerre.]

35. Prove VII. **36.** Prove VIII.

37. If $u_1 \equiv a_1x+b_1y+c_1$, $u_2 \equiv a_2x+b_2y+c_2$, prove that the pair of lines $u_1-\lambda k u_2=0$, $u_1-\lambda u_2=0$ generate homographic pencils, if λ varies; and that $u_1=0$, $u_2=0$ are the double rays.

38. Enunciate the dual of Ex. 37, by using line coordinates.

39. Use Ex. 37 to prove Theorem 143.

40. $y=mx$, $y-\beta=m'x$ are two variable lines OP, AP, passing through the origin and the fixed point $(0, \beta)$ respectively. If they make a constant angle with each other, prove that OP, AP generate homographic pencils, having the isotropic lines in each pencil as corresponding rays. Find the equation of the locus of P; and interpret these results geometrically.

41. Two homographic pencils, whose vertices O, A are the points $(0, 0)$ and $(0, \beta)$, are defined by the relation $pMm+qM+rm+s=0$. If a variable pair of corresponding rays meet at P, prove that the equation of the locus of P is $py^2+(q+r)xy+sx^2-p\beta y-r\beta x=0$, which represents a curve of the second degree, passing through O, A.

GEOMETRICAL TREATMENT OF HOMOGRAPHIC RANGES.

THEOREM 112.

A, B, C are three fixed points on a base l; A', B', C' are three other fixed points on a base l'. It is possible to construct in one and only one way, pairs of points P, P'; Q, Q'; ... on l, l' such that the cross ratio of any four of the points A, B, C, ... P, Q, ... is equal to that of the corresponding points of the range A', B', C', ... P', Q', ...

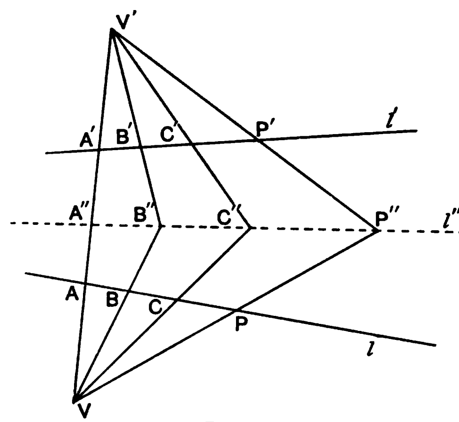

FIG. 104.

Take any two points V, V' on AA'.

B'', C'' are the meets of VB, $V'B'$; VC, $V'C'$; and l'' is the join of B'', C''.

Take any point P on l; let P'' be the meet of VP and l'', and P' the meet of $V'P''$ and l'.

Similarly, if Q, R, S, ... are any points on l, construct the corresponding points Q'', R'', S'', ... on l'' and Q', R', S', ... on l'.

Then $\{PQRS\} = V\{PQRS\} = \{P''Q''R''S''\} = V'\{P''Q''R''S''\}$
$= \{P'Q'R'S'\}$.

The construction is therefore always possible.

Again, $\{A'B'C'P'\} = \{ABCP\}$, or $\dfrac{A'B' \cdot C'P'}{A'P' \cdot C'B'} = \dfrac{AB \cdot CP}{AP \cdot CB}$.

If then the position of P is given, $\dfrac{C'P'}{A'P'}$ is fixed; and therefore there is one and only one position of P'.

Hence the construction is possible in one and only one way.

Q.E.D.

GEOMETRICAL TREATMENT OF HOMOGRAPHIC PENCILS.

THEOREM 113.

a, b, c are three fixed lines through a vertex L; a', b', c' are three other fixed lines through a vertex L'. It is possible to construct in one and only one way, pairs of lines p, p'; q, q'; ... through L, L', such that the cross ratio of any four of the lines a, b, c, ... p, q, ... is equal to that of the corresponding lines of the pencil a', b', c', ... p', q' ....

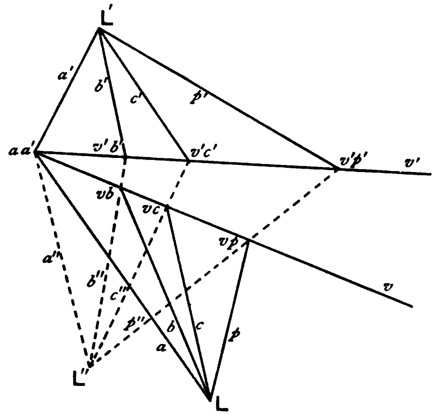

FIG. 105.

Take any two lines v, v' through aa'.

b'', c'' are the joins of vb, $v'b'$; vc, $v'c'$; and L'' is the meet of b'', c''.

Take any line p through L; let p'' be the join of vp and L'', and p' the join of $v'p''$ and L'.

Similarly, if q, r, s, ... are any lines through L, construct the corresponding lines q'', r'', s'', ... through L'' and q', r', s', ... through L'.

Then $$\{pqrs\} = v\{pqrs\} = \{p''q''r''s''\} = v'\{p''q''r''s''\} = \{p'q'r's'\}.$$

The construction is therefore always possible.

Again, $\{a'b'c'p'\} = \{abcp\}$ or $\dfrac{\sin \widehat{a'b'} \cdot \sin \widehat{c'p'}}{\sin \widehat{a'p'} \cdot \sin \widehat{c'b'}} = \dfrac{\sin \widehat{ab} \cdot \sin \widehat{cp}}{\sin \widehat{ap} \cdot \sin \widehat{cb}}$.

If then the position of p is given, $\dfrac{\sin \widehat{c'p'}}{\sin \widehat{a'p'}}$ is fixed: and therefore there is one and only one position of p'.

Hence the construction is possible in one and only one way.

Q.E.D.

THEOREM 114.

If two ranges $\{A, B, \dots P, \dots\}$, $\{A', B', \dots P', \dots\}$ are each homographic to a third range $\{A_1, B_1, \dots P_1, \dots\}$, then they are homographic to each other.

For $\{PQRS\} = \{P_1Q_1R_1S_1\}$ and $\{P'Q'R'S'\} = \{P_1Q_1R_1S_1\}$;

$$\therefore \{PQRS\} = \{P'Q'R'S'\};$$

$\therefore$ the ranges $\{A, \dots P, \dots\}$, $\{A', \dots P', \dots\}$ are homographic.

Q.E.D.

THEOREM 116.

Homographic ranges reciprocate into homographic pencils.

The proof is left to the reader: use Theorem 96 (2).

If the bases l, l' coincide, Theorem 112 is still true, for it is only necessary to take an auxiliary base l_1 (for preference through A), and to construct on it a range $A_1, B_1, C_1, \dots P_1, \dots$ homographic to the range $A, B, C, \dots P, \dots$ and then to construct on l' the range $A', B', C', \dots$ homographic to $A_1, B_1, C_1, \dots$.

Using the definition of two homographic ranges (pages 207–8), Theorem 112 may be stated as follows: **Homographic ranges on the same or different bases exist and are determined uniquely, when three given points on one base correspond to three given points on the other base.**

Definitions.

(1) Two ranges are said to be in **perspective**, if the joins of corresponding points are concurrent.

(2) Two ranges (A), (A') are said to be **projective**, if one or more ranges (A_1), (A_2), (A_3), ... can be found, such that each of the ranges (A), (A_1), (A_2), (A_3), ... (A') is in perspective with the range following it in the sequence.

THEOREM 115.

If two pencils $\{a, b, \dots p, \dots\}$, $\{a', b', \dots p', \dots\}$ are each homographic to a third pencil $\{a_1, b_1, \dots p_1, \dots\}$, then they are homographic to each other,

For $\{pqrs\} = \{p_1q_1r_1s_1\}$ and $\{p'q'r's'\} = \{p_1q_1r_1s_1\}$;

$$\therefore \{pqrs\} = \{p'q'r's'\};$$

$\therefore$ the ranges $\{a, \dots p, \dots\}$, $\{a', \dots p', \dots\}$ are homographic.

Q.E.D.

THEOREM 117.

Homographic pencils reciprocate into homographic ranges.

The proof is left to the reader.

If the vertices L, L' coincide, Theorem 113 is still true, for it is only necessary to take an auxiliary vertex L_1 (for preference on a), and to construct through it a pencil $a_1, b_1, c_1, \dots p_1, \dots$ homographic to the pencil $a, b, c, \dots p, \dots$ and then to construct through L' the pencil $a', b', c', \dots$ homographic to $a_1, b_1, c_1, \dots$.

Using the definition of two homographic pencils (page 213), Theorem 113 may be stated as follows: **Homographic pencils with the same or different vertices exist and are determined uniquely, when three given lines through one vertex correspond to three given lines through the other vertex.**

Definitions.

(1) Two pencils are said to be in **perspective**, if the meets of corresponding rays are collinear.

(2) Two pencils (a), (a') are said to be **projective**, if one or more pencils (a_1), (a_2), $(a_3) \dots$ can be found, such that each of the pencils (a), (a_1), (a_2), $(a_3) \dots (a')$ is in perspective with the pencil following it in the sequence.

THEOREM 118.

Projective ranges are homographic: and conversely homographic ranges are projective.

The proof is left to the reader.

[Use Theorem 112. If l, l' are distinct, only one auxiliary range (A_1) is required: if however l' coincides with l, two auxiliary ranges (A_1), (A_2) are needed.]

THEOREM 120.

If two homographic ranges on different bases have one self-corresponding point, they are in perspective.

The proof is left to the reader.

[Use (3) on page 206.]

THEOREM 122.

If two homographic ranges

$$\{A, B, \dots P, Q, \dots\}, \ \{A', B', \dots P', Q', \dots\}$$

have different bases l, l'; then the meet of PQ', $P'Q$ lies on a fixed straight line.

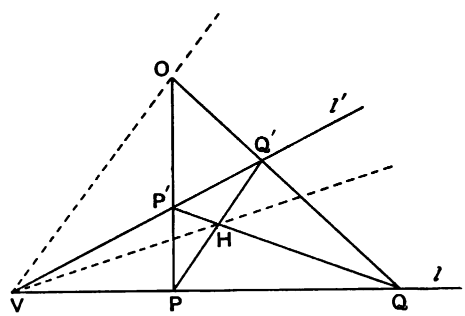

FIG. 106.

Case I. If the ranges are in perspective, let AA', BB', ... PP', ... meet at the point O, and let V be the meet of l, l' and H the meet of PQ', $P'Q$.

By the harmonic property of the quadrilateral, $V\{PP'; OH\}$ is a harmonic pencil.

But VP, VO, VP' are fixed lines;

$\therefore$ VH is a fixed line;

$\therefore$ the meet of PQ', $P'Q$ lies on a fixed line. Q.E.D.

THEOREM 119.

Projective pencils are homographic: and conversely homographic pencils are projective.

The proof is left to the reader.

[Use Theorem 113. If L, L' are distinct, only one auxiliary pencil (a_1) is required; if however L' coincides with L, two auxiliary ranges (a_1), (a_2) are needed.]

THEOREM 121.

If two homographic pencils with different vertices have one self-corresponding ray, they are in perspective.

The proof is left to the reader.

[Use (3) on page 206.]

THEOREM 123.

If two homographic pencils $\{a, b, \ldots p, q, \ldots\}$, $\{a', b', \ldots p', q', \ldots\}$ have different vertices L, L'; then the join of pq', $p'q$ passes through a fixed point.

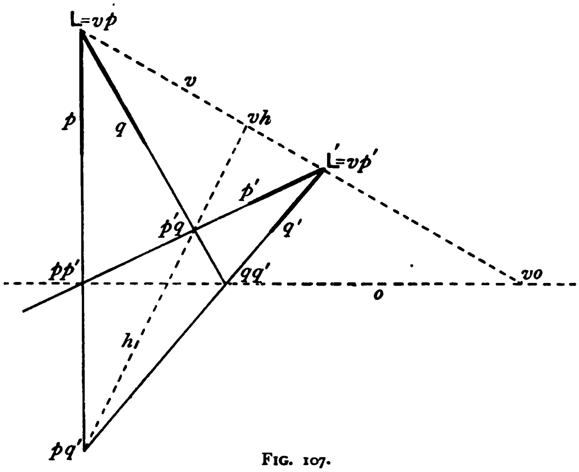

Fig. 107.

Case I. If the pencils are in perspective, let aa', bb', $\ldots pp'$, $\ldots$ lie on the line o, and let v be the join of L, L' and h the join of pq', $p'q$.

By the harmonic property of the quadrangle, $v\{pp'; oh\}$ is a harmonic range.

But vp, vo, vp' are fixed points;

$\therefore$ vh is a fixed point;

$\therefore$ the join of pq', $p'q$ passes through a fixed point. Q.E.D.

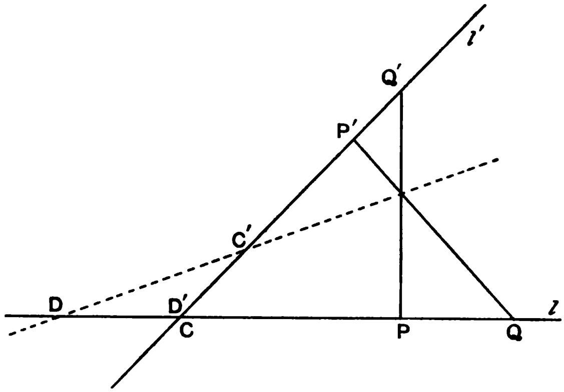

FIG. 108.

Case II. If the ranges are not in perspective, the meet of l, l' is not a self-corresponding point in the two ranges.

Denote the meet of l, l' by C or D' according as it is regarded as belonging to (A) or (A'); C', D denote the points corresponding to C, D'.

Now $P\{P'Q'C'D'\} = P'\{PQCD\}$, by hypothesis.

These have a self-corresponding ray PP'.

$\therefore$ the meets of PQ', $P'Q$; PC', $P'C$; PD', $P'D$ are collinear.

But the meet of PC', $P'C$ is C' and the meet of PD', $P'D$ is D.

$\therefore$ the meet of PQ', $P'Q$ lies on $C'D$,

which is a fixed line. Q.E.D.

Case I. may also be proved as follows:

Since O is the meet of PP', QQ'; the meet of $P'Q$, PQ' lies on the polar of O w.r.t. the conic formed by the fixed lines $l(=PQ)$ and $l'(=P'Q')$; and therefore lies on a fixed line.

Definition.

If $\{A, \dots P, Q, \dots\}$, $\{A', \dots P', Q', \dots\}$ are two homographic ranges on different bases, the straight line which is the locus of the meet of PQ', $P'Q$ is called the **cross-axis** of the two ranges.

We follow Dr. Filon in adopting what is a more suggestive name than the term "homographic axis," which is ordinarily employed.

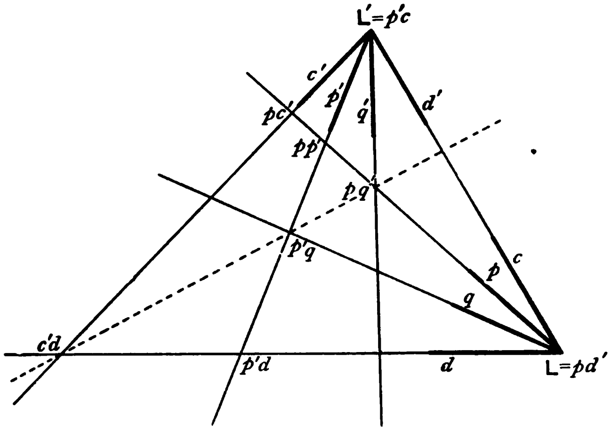

FIG. 109.

Case II. If the pencils are not in perspective, the join of L, L' is not a self-corresponding ray in the two pencils.

Denote the join of L, L' by c or d' according as it is regarded as belonging to (a) or (a'); c', d denote the rays corresponding to c, d'.

Now $p\{p'q'c'd'\} = p'\{pqcd\}$, by hypothesis.

These have a self-corresponding point pp'.

$\therefore$ the joins of pq', $p'q$; pc', $p'c$; pd', $p'd$ are concurrent.

But the join of pc', $p'c$ is c' and the join of pd', $p'd$ is d.

$\therefore$ the join of pq', $p'q$ passes through $c'd$,

which is a fixed point. Q.E.D.

Definition.

If $\{a, \ldots p, q, \ldots\}$, $\{a', \ldots p', q', \ldots\}$ are two homographic pencils with different vertices, the fixed point which lies on the join of pq', $p'q$ is called the **cross-centre** of the two pencils.

We follow Dr. Filon in adopting what is a more suggestive name than the term "homographic centre," which is ordinarily employed.

42. Prove Theorem 116.

43. Prove Theorem 117.

44. Prove Theorem 118.

45. Prove Theorem 119.

46. Prove Theorem 120.

47. Prove Theorem 121.

48. A, B are fixed points; P is a variable point on a fixed line; AP, BP meet another fixed line in P_1, P_2; prove that P_1, P_2 generate homographic ranges, and find two positions in which P_1, P_2 coincide.

49. $\{A, B, \ldots P, \ldots\}$, $\{A', B', \ldots P', \ldots\}$ are two homographic ranges on different bases; O, O' are any two fixed points on AA'; find the locus of the meet of OP, $O'P'$.

50. $O\{A, B, C, \ldots P, \ldots\}$, $O'\{A, B', C', \ldots P', \ldots\}$ are two homographic pencils; any two fixed lines AX, AX' cut OP, $O'P'$ at P, P'; prove that PP' passes through a fixed point.

51. The sides QR, RP, PQ of a variable triangle pass through fixed points A, B, C; P, Q move on fixed straight lines; prove that AR, BR generate homographic pencils.

52. P is a variable point on the common base of two homographic ranges $\{A_1, B_1, \ldots\}$, $\{A_2, B_2 \ldots\}$; P_2 and P_1 are the points corresponding to P according as it is regarded as belonging to the first or second range; prove that P_1 and P_2 generate homographic ranges.

53. A, B are two fixed points; a line through A meets two fixed lines at P_1, P_2; prove that BP_1, BP_2 generate homographic pencils, and determine the two positions in which BP_1, BP_2 coincide.

54. A, B, C; A', B', C' are two sets of three collinear points; prove that the meets of AB', $A'B$; BC', $B'C$; CA', $C'A$ are collinear.

55. If in Ex. 54, AA', BB', CC' concur at O, and if BC', $B'C$ meet at A_1, prove that OA_1 is divided harmonically by AB, $A'B'$.

56. A, B, H, K are four fixed points; X is a variable point on HK; AX, BX cut a fixed line at A', B'; HA', KB' meet at Y; prove that the locus of Y is a straight line, concurrent with AB, $A'B'$.

57. P, P' are two variable points on a fixed line l such that the circle through O, P, P' cuts l at a fixed angle, O being a fixed point; prove that P, P' generate homographic ranges.

58. PQR is a variable triangle of given shape inscribed in a fixed triangle; prove that its vertices generate homographic ranges on the sides of the fixed triangle.

59. A variable conic touches the sides AB, AC of a given triangle ABC and cuts BC at two fixed points; prove that its points of contact with AB, AC generate homographic ranges. [Use projection.]

60. A variable conic touches four fixed lines; prove that its points of contact generate homographic ranges. [Project the dual property.]

61. OA, OB, OC are three fixed lines; F, G, H are fixed points on OC; a variable line is drawn through H cutting OA, OB in P, Q; X, Y are the meets of PF, QG and PG, QF respectively; prove that X, Y move on fixed lines through O and that $O\{XPYQ\} = \{FOGH\}^2$.

62. D is a fixed point on the base BC of the triangle ABC; prove that the diagonals of the quadrilateral formed by two fixed lines through D and two variable lines through B, C, which meet on AD, pass through fixed points on BC.

DOUBLE POINTS.

Notation.

If $\{A, B, \ldots P, \ldots\}$, $\{A', B', \ldots P', \ldots\}$ are two homographic ranges on the same or different bases l, l'; we shall denote the points at infinity on l, l' by J, I' and the points corresponding to them on l', l by J', I.

If l, l' coincide, E, F will denote the double points (if they exist) of the two ranges.

THEOREM 124.

(1) If P, P' are a variable pair of corresponding points of the two homographic ranges (A), (A'), then $PI \cdot P'J'$ is constant.

(2) Conversely, if I, J' are fixed points on the fixed lines l, l', and if P, P' are a variable pair of points on l, l' respectively, such that $PI \cdot P'J'$ is constant; then P, P' generate homographic ranges.

(1) By hypothesis, $\{PIQJ\} = \{P'I'Q'J'\}$;

$$\therefore \frac{PI \cdot QJ}{PJ \cdot QI} = \frac{P'I' \cdot Q'J'}{P'J' \cdot Q'I'}.$$

But $\dfrac{QJ}{PJ} = 1 = \dfrac{P'I'}{Q'I'}$, since J, I' are points at infinity.

$$\therefore \frac{PI}{QI} = \frac{Q'J'}{P'J'};$$

$$\therefore PI \cdot P'J' = QI \cdot Q'J';$$

$$\therefore PI \cdot P'J' \text{ is constant.} \qquad \text{Q.E.D.}$$

(2) To prove the converse, it is only necessary to reverse the order of argument in (1). Analytically, the theorem is almost self-evident.

Corollary.

If l, l' coincide, and if E is a point on l such that

$$EI \cdot EJ' = AI \cdot A'J',$$

then E is a point which corresponds to itself in the two ranges, *i.e.* is a double point.

THEOREM 125.

If the lines IA, $J'A'$ are rotated in opposite directions about I, J' through a right angle, into the positions Ia, $J'a'$; then the circle on aa' as diameter cuts the common base at two points E, F (real, coincident or imaginary), which are self-corresponding points for the two ranges.

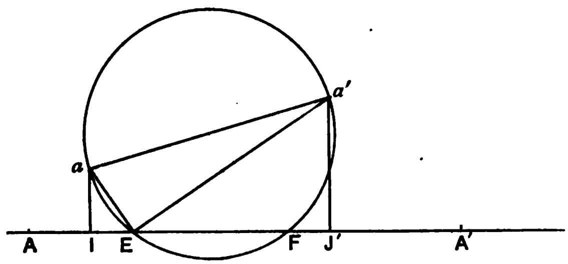

FIG. 110.

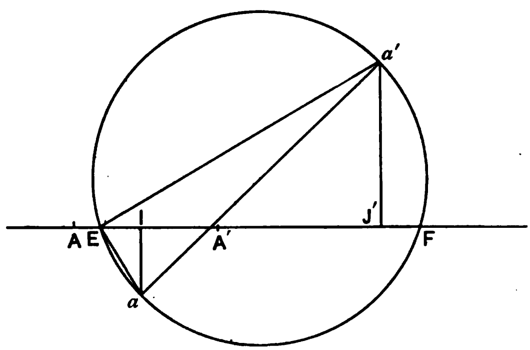

FIG. 111.

Join Ea, Ea';

Since aa' is a diameter, $a\hat{E}a' = 90°$;

$\therefore$ the triangles IaE, $J'Ea'$ are similar;

$$\therefore \frac{EI}{Ia} = \frac{a'J'}{EJ'};$$

$$\therefore EI \,.\, EJ' = Ia \,.\, a'J'$$
$$= AI \,.\, A'J', \text{ by hypothesis.}$$

Therefore by Theorem 124 Corollary, E is a double point of the two ranges.

Similarly F is also a double point.

If the circle does not meet the base at real points, the above proof breaks down, but, by the Principle of Continuity, it follows that the intersection of the base and circle yields two imaginary double points. Q.E.D.

This simple and ingenious construction for the double points of two homographic ranges on the same base is due to Professor A. Lodge. In order to carry it out, the positions of I and J' must be first determined. We shall therefore next show how to obtain I, J' if three pairs of corresponding points are given.

Construction for I, J'.

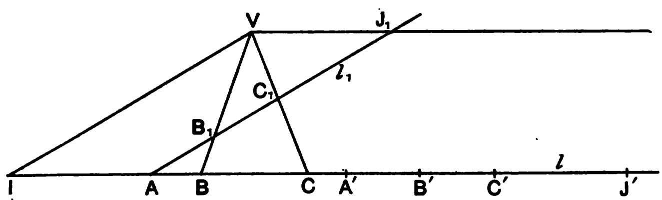

FIG. 112.

Let A, A'; B, B'; C, C' be the given pairs of corresponding points on l.

Draw through A any line l_1, and cut off AB_1, AC_1 equal to $A'B'$, $A'C'$.

Let BB_1, CC_1 meet at V.

Draw VJ_1, VI parallel to l, l_1 respectively, to meet l_1, l at J_1, I.

Cut off $A'J'$ on l equal to AJ_1; then I and J' are the required points.

The proof is left to the reader. Q.E.F.

Professor Lodge has also pointed out how simply Figures 110, 111 yield a method of generating the homographic ranges, if I, J' and a pair of corresponding points are given.

Take any point π on the circle in Fig. 110 or Fig. 111.

Let $a\pi$, $a'\pi$ meet the base in P, P'.

Then it is easy to see that the triangles aIP, $P'J'a'$ are similar;

$$\therefore \frac{PI}{aI} = \frac{J'a'}{P'J'};$$

$$\therefore PI \,.\, P'J' = aI \,.\, J'a' = AI \,.\, A'J';$$

$\therefore$ P, P' are a pair of corresponding points. Q.E.F.

THEOREM 126.

If E, F are the double points of the cobasal homographic ranges $\{A, B, \ldots P, Q, \ldots\}$, $\{A', B', \ldots P', Q', \ldots\}$, then $\{PEP'F\}$ is constant.

By hypothesis, $\{PEQF\} = \{P'EQ'F\}$;

$$\therefore \frac{PE.QF}{PF.QE} = \frac{P'E.Q'F}{P'F.Q'E};$$

$$\therefore \frac{PE.P'F}{PF.P'E} = \frac{QE.Q'F}{QF.Q'E};$$

$$\therefore \{PEP'F\} = \{QEQ'F\};$$

$\therefore$ $\{PEP'F\}$ is constant. Q.E.D.

Corollary.

If E, F are two fixed points, and if P, P' are a variable pair of points on EF, such that $\{PEP'F\}$ is constant, then P, P' generate homographic ranges, having E, F as double points.

It is only necessary to reverse the order of the argument in the proof of Theorem 126 to see this.

THEOREM 127.

If the points at infinity on the bases l, l' of the homographic ranges $\{A, B, \ldots P, \ldots\}$, $\{A', B', \ldots P', \ldots\}$ correspond, then the ranges are similar.

By hypothesis, $\{PQR\infty\} = \{P'Q'R'\infty'\}$;

$$\therefore \frac{PQ.R\infty}{P\infty.RQ} = \frac{P'Q'.R'\infty'}{P'\infty'.R'Q'}.$$

$$\text{But } \frac{R\infty}{P\infty} = 1 = \frac{R'\infty'}{P'\infty'};$$

$$\therefore \frac{PQ}{RQ} = \frac{P'Q'}{R'Q'};$$

$$\therefore \frac{PQ}{P'Q'} = \frac{QR}{Q'R'};$$

$\therefore$ the ranges are similar. Q.E.D.

Corollary. If two ranges are similar, the points at infinity on their bases correspond to each other.

THEOREM 128.

Two homographic pencils with the same vertex have two double rays, real, coincident or imaginary.

The proof is left to the reader.

[Consider the homographic ranges formed by the pencils on any transversal.]

THEOREM 129.

If e, f are the double rays of the homographic pencils $\{a, b, \ldots p \ldots\}$, $\{a', b', \ldots p' \ldots\}$, which have a common vertex, then $\{pep'f\}$ is constant.

And conversely, if e, f are two fixed lines and p, p' a variable pair of lines through ef, such that $\{pep'f\}$ is constant, then p, p' generate homographic pencils, having e, f as double rays.

The proof is left to the reader.

[Consider the ranges formed on any transversal.]

63. Prove Theorem 124(2).

64. Prove the Construction for I, J' on page 227.

65. Prove Theorem 126 Corollary.

66. Prove Theorem 127 Corollary.

67. Prove Theorem 128.

68. Prove Theorem 129.

69. Determine by a ruler and compass construction the positions of (1) I, J'; (2) E, F for the cobasal homographic ranges, defined by the pairs of points $x=0$, $x'=\frac{3}{2}$; $x=6$, $x'=0$; $x=-2$, $x'=\frac{4}{3}$. Compare your results with the positions determined by analysis.

70. If E is at infinity, prove that the ranges are similar.

71. IJ′ is a diameter of a circle, centre O; a variable tangent to the circle cuts the tangents at I and J′ to the circle in P, P′; prove that (1) the triangles OIP, P′J′O are similar; (2) IP . J′P′ is constant; (3) P, P′ trace out homographic ranges.

72. With the usual notation, prove that $EI=J'F$.

73. With the usual notation, if O′ is the point of the (A′) range which corresponds to the mid-point O of IJ′, regarded as belonging to the (A) range, prove that $OE^2=OF^2=IO \cdot OO'=OJ' \cdot OO'$. Deduce that the double points are real, if O does not lie between J′ and O′.

74. Deduce from Ex. 73 another method of constructing the double points.

THEOREM 130.

If $\{A, B, \dots\}$, $\{A', B' \dots\}$ are two homographic cobasal ranges, there exist two positions of a point L, such that the pencils $L\{A, B, \dots\}$, $L\{A', B', \dots\}$ can be superposed; *i.e.* corresponding rays are inclined to each other at a constant angle; and these two positions of L are real, if, and only if, the double points of the ranges are imaginary.

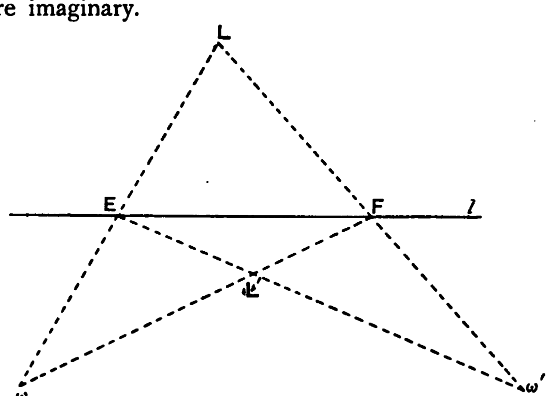

FIG. 113.

Let E, F be the double points, and denote by ω, ω' the circular points at infinity.

Let $E\omega$, $F\omega'$ meet at L and $E\omega'$, $F\omega$ meet at L'.

Then the isotropic lines are the double rays of the homographic pencils $L\{A, B, \dots\}$, $L\{A', B', \dots\}$.

∴ corresponding rays make a constant angle with each other. [Th. 11, and Th. 129.]

∴ L is one of the required points; and similarly L' is another possible position.

Further, if E, F are imaginary points, they are conjugate imaginaries, and therefore the lines $E\omega$, $F\omega'$ are conjugate imaginary lines.

∴ their meet L, and similarly L', are real points.

But if E, F are real, there cannot be any other real point on the lines $E\omega$, $F\omega'$; and therefore L, and similarly L', are (conjugate) imaginary points. Q.E.D.

75. In Theorem 130, prove that L' is the reflection of L in the common base of the ranges.

76. Two homographic ranges on the x-axis are determined by the relation $xx' - 2ax + a^2 + b^2 = 0$, find the coordinates of the two positions of the point L of Theorem 130.

77. Repeat Ex. 76, for the relation $xx'-2ax+a^2-b^2=0$.

78. Prove that two homographic pencils with a common vertex can be projected into pencils in which corresponding rays are inclined at a constant angle.

79. A, B are two fixed points; P, P' are a pair of variable points on AB such that (1) $\frac{AP}{PB}:\frac{AP'}{P'B}$ or (2) $\lambda\frac{AP}{PB}+\mu\frac{AP'}{P'B}$ is constant; λ, μ being constants; prove that P, P' generate homographic ranges. Determine the double points in each case.

80. If two homographic pencils have different vertices, prove that there are two pairs of parallel corresponding rays.

81. Find the form of the homographic relation connecting two cobasal ranges, if their double points coincide at the origin.

82. With the usual notation, prove that $AP.J'P'$ is proportional to $A'P'$, where A, A' are fixed.

83. With the usual notation, prove that $\frac{PE}{PF}$ is proportional to $\frac{P'E}{P'F}$.

84. With the usual notation, prove that $\frac{AI}{AP}+\frac{A'J'}{A'P'}=1$.

85. If $\{A, B, \ldots P, \ldots\}$, $\{A', B', \ldots P', \ldots\}$ are two homographic ranges, prove that there are two pairs of corresponding points P, P' such that AP, $A'P'$, irrespective of sense, are equal.

86. With the usual notation, prove that

$$\frac{AP}{A'P'}.BC+\frac{BP}{B'P'}.CA+\frac{CP}{C'P'}.AB=0.$$

87. E, F are the double points of the ranges determined by $xx'-x-2x'-4=0$; the points A, B are determined by $x=0$ and $x=2$; A', B' are the corresponding points; prove that the circles whose diameters are EF, AB', $A'B$ are coaxal.

88. With the usual notation, prove that $\frac{EB.EC'}{EB'.EC}=\frac{AB.A'C'}{A'B'.AC}$; hence prove that the circles whose diameters are EF, BC', $B'C$ are coaxal.

89. Deduce from Ex. 88 a method of constructing the double points, given three pairs of corresponding points.

90. If $xx'+qx+rx'+s=0$ is the relation connecting the ranges $\{A, B, \ldots\}$, $\{A', B', \ldots\}$, referred to origins A, B' respectively; prove that $q=-B'J'$, $r=-AI$, $s=AI.B'A'$.

91. If O is the mid-point of IJ', and if O' is the point corresponding to O, prove that $\frac{OP-O'P'}{OP.OP'}$ is constant.

92. A, B are two fixed points on a circle Σ; P is a variable point on Σ; prove that AP, BP generate homographic pencils. Determine

the ray in the pencil, vertex A, corresponding to the ray BA in the pencil, vertex B.

What does Theorem 123 give when applied to this result?

93. If in Ex. 92, any transversal L is drawn cutting the circle at real points and cutting AP, BP at P_1, P_2; determine the double points of the homographic ranges generated by P_1, P_2 on L.

What is the condition that the double points (1) coincide, (2) are imaginary?

94. Deduce, from Ex. 93, that every straight line meets a circle at two points (real, coincident, or imaginary).

95. A variable tangent to a fixed circle, centre O, cuts two fixed tangents to the circle at P_1, P_2; prove that (1) $P_1\hat{O}P_2$ is constant; (2) P_1 and P_2 generate homographic ranges.

What points in these ranges correspond to the meet of the two fixed tangents?

96. With the notation of Ex. 95, if any point L is taken outside the circle, determine the double rays of the homographic pencils generated by LP_1, LP_2.

97. Deduce, from Ex. 96, that from every point two tangents (real, coincident, or imaginary) can be drawn to a circle.

98. If one double point of the cobasal homographic ranges $\{A, B, \ldots\}$, $\{A', B', \ldots\}$ is at infinity, prove that the radical axis of the circles, whose diameters are AB', $A'B$, passes through the other double point.

99. Given three segments AA', BB', CC' on a straight line, determine a point O, such that $A\hat{O}A' = B\hat{O}B' = C\hat{O}C'$.

100. With the usual notation, prove that $\dfrac{EI}{EP} + \dfrac{EJ'}{EP'} = 1$.

101. With the usual notation, prove that

$$\frac{AB \cdot CD}{A'B'} + \frac{AC \cdot DB}{A'C'} + \frac{AD \cdot BC}{A'D'} = 0.$$

102. The double points of two cobasal homographic ranges coincide at E; A_1 is any point on the base; A, A' correspond to A_1 when regarded as belonging to the first and second range respectively; prove that $\{EA_1; AA'\}$ is harmonic.

Two constructions for obtaining the double points of cobasal homographic ranges have been given (see page 226 and Ex. 74); a third will be found on page 244.

The following examples illustrate their use in performing a wide group of constructions.

The method is known as construction by **Trial and Error**, or the method of **false positions.**

EXAMPLE I. [PONCELET'S PROBLEM].

To describe a quadrilateral, such that its four sides pass through given points, and its four corners lie on given lines.

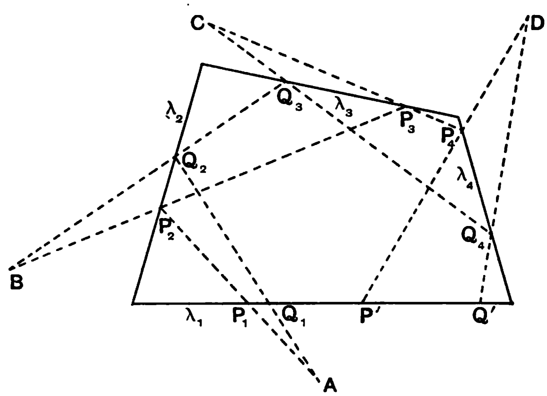

FIG. 114.

Let A, B, C, D be the four fixed points, and λ_1, λ_2, λ_3, λ_4 the four fixed lines.

Through A, draw any line cutting λ_1, λ_2 at P_1, P_2; join BP_2 and produce it to cut λ_3 at P_3; join CP_3 and produce it to cut λ_4 at P_4; join DP_4 and produce it to cut λ_1 at P'.

Similarly, take any number of lines through A and construct the sets of points Q_1, Q_2, Q_3, Q_4, Q'; R_1, R_2, R_3, R_4, R'; ....

Then

$$\begin{aligned}\{P_1, Q_1, \ldots\} &= A\{P_1, Q_1, \ldots\} = \{P_2, Q_2, \ldots\} = B\{P_2, Q_2, \ldots\}\\ &= \{P_3, Q_3, \ldots\} = C\{P_3, Q_3, \ldots\} = \{P_4, Q_4, \ldots\}\\ &= D\{P_4, Q_4, \ldots\} = \{P', Q', \ldots\};\end{aligned}$$

$\therefore$ $\{P_1, Q_1, \ldots\}$, $\{P', Q', \ldots\}$ are homographic ranges on λ_1.

Let E, F be the double points of the two ranges.

Then either AE or AF may be taken as a side of the required quadrilateral: and then the remaining sides are at once determined.

Q.E.F.

The method is general, and applies with equal ease to the case of an n-sided polygon, whose sides pass through fixed points and whose n-corners lie on fixed lines.

The practical application of this method will provide the reader with an exercise in drawing of by no means a simple character. We give below the *complete* construction for Poncelet's Problem in the case of a triangle.

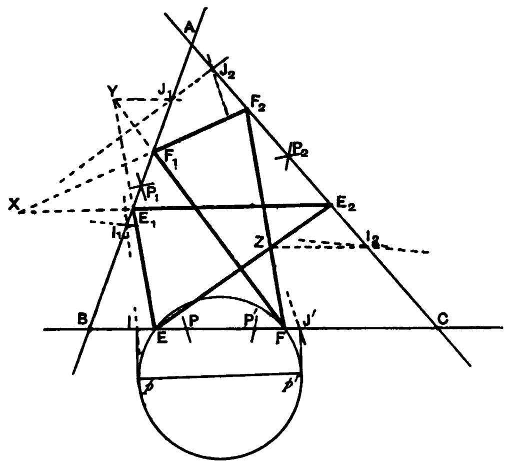

FIG. 115.

ABC is a fixed triangle; X, Y, Z are fixed points.

It is required to inscribe a triangle in ABC, such that its sides pass through X, Y, Z respectively.

P is any (convenient) point on BC; PY meets AB at P_1; XP_1 meets AC at P_2; P_2Z meets BC at P'.

To avoid unnecessary complications, only small parts of the lines PY, etc., are drawn in the figure.

We have now to determine the double points of the homographic ranges, generated by P, P'; and so we first find I, J'.

Through Y draw a line parallel to BC to cut AB at J_1; XJ_1 cuts AC at J_2; J_2Z cuts BC at the required point J'.

Through Z draw a line parallel to BC to cut AC at I_2; XI_2 cuts AB at I_1; I_1Y cuts BC at the required point I.

Rotate the lines IP, $J'P'$ in opposite directions through a right angle into the positions Ip, $J'p'$.

On pp' as diameter describe a circle cutting BC at E, F, so that E, F are the required double points.

Let EY, FY cut AB at E_1, F_1 and let E_1X, F_1X cut AC at E_2, F_2; then E_2Z, F_2Z must cut BC at E, F.

$\therefore$ EE_1E_2 and FF_1F_2 are the two triangles, which satisfy the given conditions. Q.E.F.

To acquire familiarity with the nature of the construction, the reader is advised to attempt this problem for himself.

EXAMPLE II.

To inscribe in a given triangle a rectangle of given area.

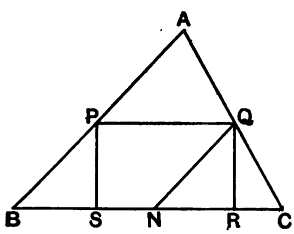

FIG. 116.

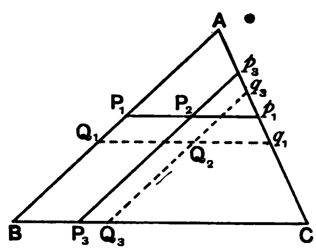

FIG. 117.

If $PQRS$ is the required rectangle, draw QN parallel to AB, then $PQNB$ is a parallelogram equal in area to the rectangle.

Hence it is only necessary to determine a point Q on AC, such that the parallelogram, formed by BA, BC and lines through Q parallel to BC, BA, is of given area.

Take any point P_1 on BA and cut off BP_3 from BC so that the completed parallelogram $P_3BP_1P_2$ is of the given area, and produce P_1P_2, P_3P_2 to meet AC at p_1, p_3.

Similarly construct the sets of points Q_1, Q_2, Q_3, q_1, q_3; R_1, R_2, R_3, r_1, r_3, ...

By hypothesis $BP_1 \,.\, BP_3 = \text{constant} = BQ_1 \,.\, BQ_3 = \ldots$;

$\therefore$ the ranges $\{P_1, Q_1, \ldots\}$, $\{P_3, Q_3, \ldots\}$ are homographic;

$\therefore$ by parallels, the ranges $\{p_1, q_1, \ldots\}$, $\{p_3, q_3, \ldots\}$ are homographic.

Let e, f be their double points.

Then either e or f can be taken as a vertex of the required rectangle. Q.E.F.

103. $\{A, B, \ldots P, \ldots\}$, $\{A', B', \ldots P', \ldots\}$ are two homographic ranges on different bases l, l'; through a given point, draw a line to cut l and l' in corresponding points. How many solutions are there?

104. With the notation of Ex. 103, if α, β are two given points, draw two lines αP, $\beta P'$ to intersect on a given circle through α, β.

105. With the notation of Ex. 103, find the positions of P, P' such that PP' subtends a given angle at a given point.

106. A, B are fixed points on the fixed lines l, m; through a given point O, draw a line to cut l, m at P, Q such that $AP . BQ$ has a given value.

107. Through a given point, draw a line to cut two given lines at points subtending a given angle at a given point.

108. Find a segment of a given line, which subtends angles of given size at each of two given points.

109. Find two points P, Q on two fixed lines, such that PQ subtends angles of given size at each of two fixed points.

110. Given three fixed lines l, l', m, find two points P, P' on l, l' respectively, such that PP' subtends a given angle at a given point and has a projection of given length on m.

111. Given four fixed lines l, l', m, m', construct a line cutting l, l' at P, P', such that the projections of PP' on m, m' are of given lengths.

112. Find a point D in the base BC of a triangle ABC, such that the incircles of ABD, ACD may touch AD at the same point.

113. With the notation of Ex. 103, if α, β are two fixed points, construct points P, P'; Q, Q' such that the angles $P\alpha Q$, $P'\beta Q'$ are of given size.

114. With the notation of Ex. 103, construct points P, P'; Q, Q' such that PQ, $P'Q'$ are of given lengths.

115. α, β are fixed points on the common base of two homographic ranges $\{A, B, \dots P, \dots\}$, $\{A', B', \dots P', \dots\}$; construct the pair of points P, P', such that $\{\alpha\beta; PP'\}$ is harmonic.

116. Through a given point draw two lines to cut off, on two given lines, segments of given lengths.

117. $\{A, B, \dots\}$, $\{A', B', \dots\}$ and $\{a, b, \dots\}$, $\{a', b', \dots\}$ are two pairs of homographic ranges on four distinct bases. Determine the pairs of points P, P'; q, q' such that Pq, $P'q'$ meet at a given point.

118. If in Ex. 117, the four bases are coincident, determine a pair of points α, α' which are corresponding points in both pairs of ranges.

119. Through a given point, draw two lines which cut off from two fixed lines, segments subtending given angles at given points.

120. Describe a triangle PQR, so that its sides pass through given points, P, Q lie on fixed lines, and $P\hat{R}Q$ is of given magnitude.

121. Generalise Ex. 120 for an n-sided polygon.

122. A, B are fixed points; l, m are fixed lines; find a point P on l such that PA, PB cut off from m a segment of given length.

123. A ray of light starts from a given source and is reflected successively at n lines; if its final path makes a given angle with its initial path, construct its initial path.

124. Given a triangle ABC, a point D on BC and a point E on AD, find the points of contact with AB, AC of a conic inscribed in the triangle ABC and passing through D, E.

125. Describe a quadrilateral, so that its four corners lie on fixed lines, two of its sides pass through fixed points and the other two are in a given direction.

126. Show how to determine a pair of parallel straight lines, which pass through fixed points, and cut two fixed lines in points collinear with a given point.

CHAPTER IX.

HOMOGRAPHIC PROPERTIES OF THE CONIC.

V_1, V_2; A, B, C, D, E, ... are a system of points on a conic.

By the fundamental cross ratio property, the pencils

$$V_1\{A, B, C, \ldots\}, \quad V_2\{A, B, C, \ldots\}$$

are homographic.

It is therefore unnecessary to specify the particular position of the point V on the conic, when dealing with cross ratio properties of the pencil $V\{A, B, C, \ldots\}$.

Definitions.

(1) A system of points A, B, ... on a conic is called a **range of points on the conic** or a **range of the second order.**

(2) If two ranges of points on a conic, A, B, ... ; A', B', ... are such that the pencils $V\{A, B, \ldots\}$, $V\{A', B', \ldots\}$ are homographic, V being any point on the conic, the ranges are said to be **homographic.**

THEOREM 131.

Two homographic ranges of points on a conic exist and are determined uniquely, when three pairs of corresponding points are given.

Let A, A'; B, B'; C, C' be the given pairs of points; and let V be any other point on the conic.

Then by Theorem 113, two homographic pencils exist, and are determined uniquely, by the pairs of rays VA, VA'; VB, VB'; VC, VC'.

Let the other points of intersection of any other pair of rays VP, VP' of these pencils with the conic be P, P'.

Then P, P' are a pair of corresponding points on the conic.

Q.E.D.

CHAPTER IX.

HOMOGRAPHIC PROPERTIES OF THE CONIC.

v_1, v_2; $a, b, c, d, e, \ldots$ are a system of tangents to a conic.

By the fundamental cross ratio property, the ranges

$$v_1\{a, b, c, \ldots\},\ v_2\{a, b, c, \ldots\}$$

are homographic.

It is therefore unnecessary to specify the particular position of the tangent v to the conic, when dealing with cross ratio properties of the range $v\{a, b, c, \ldots\}$.

Definitions.

(1) A system of tangents $a, b, \ldots$ to a conic is called a **pencil of tangents to the conic** or a **pencil of the second order.**

(2) If two pencils of tangents to a conic, $a, b, \ldots$; $a', b', \ldots$ are such that the ranges $v\{a, b, \ldots\}$, $v\{a', b', \ldots\}$ are homographic, v being any tangent to the conic, the pencils are said to be **homographic.**

THEOREM 132.

Two homographic pencils of tangents to a conic exist and are determined uniquely, when three pairs of corresponding tangents are given.

Let a, a'; b, b'; c, c' be the given pairs of tangents; and let v be any other tangent to the conic.

Then by Theorem 112, two homographic ranges exist, and are determined uniquely, by the pairs of points va, va'; vb, vb'; vc, vc'.

Let the other tangents from any other pair of points vp, vp' of these ranges to the conic be p, p'.

Then p, p' are a pair of corresponding tangents to the conic.

Q.E.D.

THEOREM 133.

(1) If $\{A, B, \ldots P, Q, \ldots\}$, $\{A', B', \ldots P', Q', \ldots\}$ are two homographic ranges of points on a conic, there exist two points E, F, (real, coincident, or conjugate imaginaries) on the conic, which are self-corresponding for the two ranges.

(2) The meet of PQ', $P'Q$ lies on the fixed line EF.

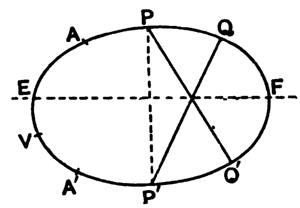

FIG. 118.

(1) Take any point V on the conic.

By Theorem 128, the pencils $V\{A, B, \ldots\}$, $V\{A', B', \ldots\}$ have two double rays.

Let the other points of intersection of these double rays with the conic be E, F.

Then E, F are clearly self-corresponding points for the two ranges of points on the conic. Q.E.D.

(2) Since E, F are self-corresponding points,

$$P\{P'EFQ'\} = P'\{PEFQ\}.$$

These two pencils have a self-corresponding ray PP'.

$\therefore$ the meets of PE, $P'E$; PF, $P'F$; PQ', $P'Q$ are collinear.

$\therefore$ the meet of PQ', $P'Q$ lies on the fixed line EF.

This fixed line is necessarily real, since

$$A, B, \ldots P \ldots, A', B', \ldots P'$$

are assumed real; and therefore its points of intersection E, F, with the conic are either real, coincident, or conjugate imaginaries.

Q.E.D.

Definition.

With the notation of Theorem 133, E, F are called the **double points** of the two homographic ranges of points on the conic; and the line EF is called the **cross-axis** of the two homographic ranges. [See page 222.]

THEOREM 134.

(1) If $\{a, b, \ldots p, q, \ldots\}$, $\{a', b', \ldots p', q', \ldots\}$ are two homographic pencils of tangents to a conic, there exist two tangents e, f, (real, coincident, or conjugate imaginaries) to the conic, which are self-corresponding for the two pencils.

(2) The join of pq', $p'q$ passes through the fixed point ef.

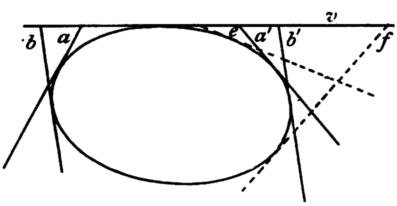

FIG. 119.

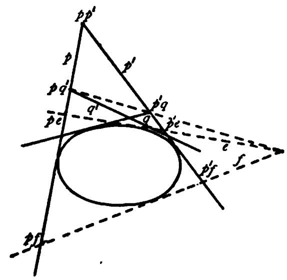

FIG. 120.

(1) Take any tangent v to the conic.

By Theorem 125, the ranges $v\{a, b, \ldots\}$, $v\{a', b', \ldots\}$ have two double points.

Let the other tangents from these double points to the conic be e, f.

Then e, f are clearly self-corresponding rays for the two pencils of tangents to the conic. Q.E.D.

(2) Since e, f are self-corresponding rays,

$$p\{p'efq'\} = p'\{pefq\}.$$

These two ranges have a self-corresponding point pp'.

∴ the joins of pe, $p'e$; pf, $p'f$; pq', $p'q$ are concurrent.

∴ the join of pq', $p'q$ passes through the fixed point ef.

This fixed point is necessarily real, since

$$a, b, \ldots p \ldots, \; a', b', \ldots p' \ldots$$

are assumed real; and therefore the tangents e, f from it to the conic are either real, coincident, or conjugate imaginaries.

Q.E.D.

Definition.

With the notation of Theorem 134, e, f are called the **double lines** of the two homographic pencils of tangents to the conic, and the point ef is called the **cross-centre** of the two homographic pencils. [See page 223.]

It should be noted that Pascal's theorem is a special case of Theorem 133.

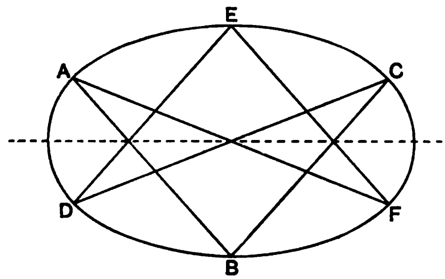

FIG. 121.

Let $ABCDEF$ be a hexagon inscribed in a conic.

Consider the homographic ranges of points on the conic, determined by $\{A, C, E\}$, $\{D, F, B\}$.

By Theorem 133, the meets of AB, DE; BC, EF; CD, FA lie on a straight line, the cross-axis; which is Pascal's property.

Q.E.D.

THEOREM 135.

Homographic ranges of points on a conic project into homographic ranges of points on the projected conic: double points project into double points: and the cross-axis into the cross-axis.

The proof is left to the reader.

THEOREM 137.

Homographic ranges of points on a conic reciprocate into homographic pencils of tangents to the reciprocal conic: double points reciprocate into double lines: and the cross-axis into the cross-centre.

The proof is left to the reader.

It should be noted that Brianchon's theorem is a special case of Theorem 134.

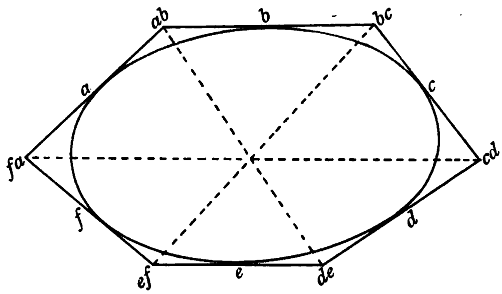

FIG. 122.

Let *abcdef* be a hexagon circumscribing a conic.

Consider the homographic pencils of tangents to the conic determined by $\{a, c, e\}$, $\{d, f, b\}$.

By Theorem 134, the joins of *ab*, *de*; *bc*, *ef*; *cd*, *fa* pass through a point, the cross-centre; which is Brianchon's property. Q.E.D.

THEOREM 136.

Homographic pencils of tangents to a conic project into homographic pencils of tangents to the projected conic: double lines project into double lines: and the cross-centre into the cross-centre.

The proof is left to the reader.

THEOREM 138.

Homographic pencils of tangents to a conic reciprocate into homographic ranges of points on the reciprocal conic: double lines reciprocate into double points: and the cross-centre into the cross-axis.

The proof is left to the reader.

THEOREM 139.

If $\{A, B, C, \ldots\}$ is a range of points on a conic, and if $\{a, b, c, \ldots\}$ is the pencil of the tangents to the conic at these points, the range $\{A, B, C, \ldots\}$ and the pencil $\{a, b, c, \ldots\}$ are homographic [*i.e.* any section of the pencil $V\{A, B, C, \ldots\}$ is homographic to the range $v\{a, b, c, \ldots\}$].

The proof is left to the reader.

[Use Theorem 57.]

THEOREM 140.

If $\{A, B, C, \ldots\}$ is a range of collinear points; and if $\{a, b, c, \ldots\}$ is the pencil of concurrent lines, formed by their polars w.r.t. any conic; the range $\{A, B, C, \ldots\}$ and the pencil $\{a, b, c, \ldots\}$ are homographic [*i.e.* any section of the pencil is homographic to the range].

The proof is left to the reader.

[Use Theorem 55.]

Double-Point Construction.

The property of the cross-axis, established in Theorem 133, affords another method of determining the double points of two cobasal homographic ranges.

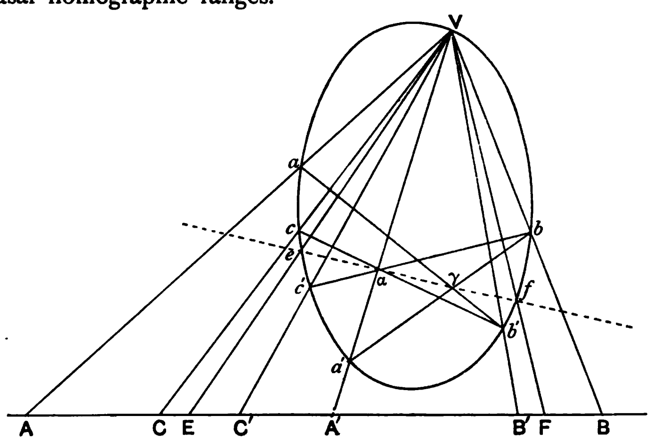

FIG. 123.

Let $\{A, B, C, \ldots\}$, $\{A', B', C' \ldots\}$ be the two ranges.

Draw any circle or conic, and take any point V on it.

Let VA, VB, VC, VA', VB', VC' cut the conic again at a, b, c, a', b', c'.

Let ab', $a'b$ meet at γ and bc', $b'c$ meet at α. Let $\alpha\gamma$ cut the conic at e, f. Join Ve, Vf, and produce them to meet the base of the given ranges at E, F.

Then E, F are the required double points.

The proof is left to the reader. Q.E.F.

[Use Theorem 133.]

1. Prove Theorem 135.

2. Prove Theorem 136.

3. Prove Theorem 137.

4. Prove Theorem 138.

5. Prove Theorem 139.

6. Prove Theorem 140.

7. Prove the Construction on page 244.

8. The base of a variable triangle inscribed in a given conic is fixed, prove that the sides generate homographic pencils.

9. One vertex of a triangle, self-conjugate w.r.t. a given conic, is fixed; prove that the other vertices generate homographic ranges.

10. AP, AP' are a variable pair of conjugate lines through a fixed point A w.r.t. a given conic; prove that they generate homographic pencils, and determine the double rays.

11. APQ is a triangle inscribed in a given conic; A is fixed and $P\hat{A}Q$ is of constant size; prove that the tangents at P, Q generate homographic pencils of tangents to the conic.

12. Two conics S_1, S_2, have double contact at B, C; A is the pole of BC; tangents are drawn from a variable point on AB to S_1, S_2 to cut AC at P_1, P_2; prove that P_1, P_2 generate homographic ranges and determine their double points.

13. l_1, l_2 are two fixed lines; P_1 is a variable point on l_1 and P_2 is a point on l_2 conjugate to P_1 w.r.t. a fixed conic; prove that P_1, P_2 generate homographic ranges.

14. The sides QR, RP, PQ of a triangle pass through fixed points A, B, C; P lies on a fixed conic through B, C; Q lies on a fixed conic through C, A; prove that BR, AR generate homographic pencils.

15. AB is a fixed chord of a circle; PQ is a variable chord of constant length; prove that AP, BQ generate homographic pencils.

16. A variable line passes through a fixed point; prove that its poles w.r.t. two given conics generate homographic ranges.

17. A is one of the common points of a system of coaxal circles; prove that any two lines through A are cut homographically by the circles. Generalise this theorem.

18. One vertex of a variable triangle circumscribing a given conic is fixed; prove that the other vertices generate homographic ranges.

19. Two conics have double contact at B, C; a variable line through B cuts the conics at P, Q; prove that CP, CQ generate homographic pencils.

20. Two variable lines are conjugate w.r.t. a given conic; if each passes through a fixed point, prove that they generate homographic pencils.

21. A, B are fixed points on a hyperbola; P is a variable point on the curve; PA, PB meet an asymptote at P_1, P_2; prove that P_1, P_2 generate homographic ranges. Where are the double points? Deduce that P_1P_2 is of constant length.

22. A, B are two fixed points on a parabola; P is a variable point on the curve; parallels to PA, PB are drawn through a fixed point O to cut a fixed diameter in P_1, P_2; prove that P_1, P_2 generate homographic ranges and that P_1P_2 is of constant length.

23. From a fixed point O, a line OP is drawn parallel to a variable tangent p of a given parabola; prove that the pencil generated by OP is homographic to the pencil of tangents to the parabola, generated by p.

24. Generalise by projection:—if a variable line through the centre of a given circle cuts the circle at P, P'; then P, P' generate homographic ranges of points on the circle.

25. Generalise by projection:—if a variable tangent to a given circle cuts a given concentric circle at P, P'; then P, P' generate homographic ranges of points on the circle.

26. Given five points A, B, C, D, E; show how to determine the points in which the conic through A, B, C, D, E cuts a given line l.
[Consider the ranges formed on l by $A\{C, D, E\}$, $B\{C, D, E\}$.]

27. Show how to determine the directions of the asymptotes of the conic passing through five given points.

28. Show how to determine the points of intersection of a given line with a conic, when three points on the conic and the directions of its asymptotes are given.

29. HK is a fixed diameter of a given conic; a variable tangent meets the tangents at H, K in P, P′; prove that HP . KP′ is constant. [Use Theorem 124.]

30. A variable tangent to a hyperbola, centre C, cuts the asymptotes at P, Q; prove that P, Q generate homographic ranges, and deduce that $CP . CQ$ is constant. [Use Theorem 124.]

31. *ABCD* is a fixed parallelogram, circumscribing a given conic; a variable tangent cuts AB, AD at P, Q; prove that $BP.DQ$ is constant. [Use Theorem 124.]

32. A variable pair of conjugate diameters of a given conic meet the tangent at a fixed point P in Q, Q′; prove that Q, Q′ generate homographic ranges, and deduce that PQ . PQ′ is constant. [Use Theorem 124.]

33. Two fixed conics S_1, S_2 meet at A, B, C, D; a variable line through A cuts S_1, S_2 at P_1, P_2; the tangents at P_1, P_2 to S_1, S_2 meet BC at Q_1, Q_2; prove that Q_1, Q_2 generate homographic ranges.

THEOREM 141.

(1) If $\{A, B, \dots P\}$, $\{A', B', \dots P'\}$ are two homographic ranges of points on a conic, the joins AA', BB', ... of corresponding points envelope a conic having double contact with the given conic at the double points E, F of the two ranges.

(2) Conversely, if two conics have double contact at E, F, a variable tangent to one cuts the other at a pair of points P, P', which generate homographic ranges of points on the conic, with E, F as double points.

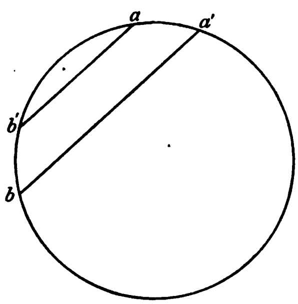

FIG. 124.

(1) Project E, F into the circular points at infinity.

Then the conic becomes a circle, and the cross-axis of the projected homographic ranges $\{a, b, \dots p, \dots\}$, $\{a', b', \dots p', \dots\}$ is the line at infinity, since EF was the former cross-axis.

$\therefore$ ab', $a'b$ are parallel.

$\therefore$ chord aa' = chord bb'.

$\therefore$ $aa' = bb' = cc' = \dots = pp' = \dots$.

$\therefore$ the lines aa', bb', cc' ... touch a concentric circle.

$\therefore$ in the original figure, AA', BB', ... envelope a conic having double contact with the given conic at E, F. Q.E.D.

(2) The proof is left to the reader.

[Project E, F into the circular points at infinity.]

Corollary.

If a variable chord PP' of a conic passes through a fixed point O, then P, P' generate homographic ranges of points on the conic, having as double points the points of contact of the tangents from O to the conic.

[Project the conic into a circle, having the projection of O as centre.]

It will appear later (page 298) that the homographic ranges in the Corollary belong to a very special type, viz., involution ranges.

THEOREM 142.

(1) If $\{a, b, \dots p, \dots\}$, $\{a', b', \dots p', \dots\}$ are two homographic pencils of tangents to a conic, the meets aa', bb', ... of corresponding tangents lie on a conic having double contact with the given conic at the points of contact with the conic of the double lines e, f of the two pencils.

(2) Conversely, if two conics have double contact, the tangents from a variable point on one, to the other, generate homographic pencils of tangents to the conic, with the tangents e, f at the points of contact as double lines.

The proof is left to the reader.

[Prove it by (1) reciprocating Theorem 141, (2) projecting the points of contact of e, f into the circular points at infinity.]

Corollary.

From a variable point P on a fixed line, tangents p_1, p_2 are drawn to a given conic, then p_1, p_2 generate homographic pencils of tangents to the conic.

[Project the conic into a circle and the fixed line to infinity.]

It will appear later (page 299) that the homographic pencils in the Corollary belong to a very special type, viz., involution pencils.

34. Prove Theorem 141 (2).

35. Prove the Corollary of Theorem 141.

36. Prove Theorem 142, by each of the methods suggested.

37. Prove the Corollary of Theorem 142.

38. H is a point of intersection of two given conics S_1, S_2; a variable line through H cuts S_1, S_2 at P_1, P_2; prove that P_1 and P_2 generate homographic ranges of points on S_1 and S_2.

39. P is a variable point on a common chord of two given conics S_1, S_2; PP_1, PP_2 are tangents from P to S_1, S_2; prove that they generate homographic pencils of tangents to S_1 and S_2.

40. A fixed circle cuts a variable circle of a given coaxal system at P, Q; prove that P, Q generate homographic ranges of points on the fixed circle. Generalise this theorem.

41. B, C are a fixed pair of conjugate points w.r.t. a given conic S; P is a variable point on S; BP, CP meet S again at P_1, P_2; prove that P_1, P_2 generate homographic ranges of points on the conic. Determine the position of the double points.

42. A variable conic passes through four fixed points; prove that the tangents at these points generate homographic pencils.

43. If $\{A, B, \ldots\}$, $\{A', B', \ldots\}$ are two homographic ranges of points on a conic; prove that the tangents at A, A'; B, B'; etc., meet the cross-axis in homographic ranges.

44. A, B are two fixed points; PAQ is a variable chord of a given conic; BP, BQ meet the conic again at P', Q'; prove that P', Q' generate homographic ranges of points on the conic.

45. A system of conics pass through four fixed points; A, B are any two other points; prove that the polars of A w.r.t. the system of conics form a pencil homographic to the polars of B w.r.t. the system.

46. AB, AC are a fixed pair of conjugate lines w.r.t. a conic S; a variable tangent to S cuts AB, AC at P, P'; prove that the other tangents from P, P' to S generate homographic pencils of tangents to S.

47. A is a fixed point; PQ is a variable chord of a given circle S; if the circle APQ cuts S at a constant angle, prove that P, Q generate homographic ranges of points on S.

48. Show how to inscribe in a given conic a triangle, such that each of its sides passes through a given point. [Use Theorem 141 Corollary.]

49. Show how to inscribe in a given conic an n-sided polygon, such that each side passes through a given point.

50. PQR is a variable triangle, inscribed in a given conic; PQ, PR pass through fixed points; find the envelope of QR.

51. $PQRS$ is a quadrilateral circumscribing a conic; P, Q, R lie on fixed lines; find the locus of S.

52. $\{A, B, \dots P \dots\}$, $\{A', B', \dots P' \dots\}$ are two homographic ranges of points on a conic; show how to determine the pair of points P, P' whose join passes through a fixed point.

53. A variable triangle is inscribed in a given conic; two of its sides are fixed in direction; find the envelope of the third side.

54. $\{a, b, \dots p \dots\}$, $\{a', b', \dots p' \dots\}$ are two homographic pencils of tangents to a conic; find the envelope of the polar of the meet of p, p'.

55. A variable triangle circumscribes a conic; two of its vertices lie on fixed lines; prove that the points of contact of its sides generate three homographic ranges of points on the conic.

56. P is a variable point on the side BC of a triangle ABC in which are inscribed two given conics S_1, S_2; PT_1, PT_2 are the other tangents from P to S_1, S_2; prove that AT_1, AT_2 generate homographic pencils: and determine the double lines.

57. Two conics S_1, S_2 have double contact with each other; the tangents from a variable point on S_1 to S_2 meet S_1 again at P, Q; prove that P, Q generate homographic ranges of points on S_1.

58. ABC is a fixed triangle inscribed in a conic; PQ is a variable chord such that $A\{BCPQ\}$ is constant; find the envelope of PQ.

59. A, B, C, D and p, q, r, s are the common points and common tangents of two conics, prove that the range of points A, B, C, D on one is equicross with the range of tangents p, q, r, s to the other.

60. Two conics S_1, S_2 have double contact at A, B; a variable chord PQ of S_1 touches S_2; find the locus of the meet of AP, BQ.

It is convenient at this stage to insert purely geometrical proofs of the important converse theorems, relating to homographic ranges and pencils, referred to on pages 116–7. The reader will note that no property is assumed in their proof, which was established after Theorem 57; but on account of their intrinsic difficulty, it seemed better to make use of the analytical results of Chapter I., to secure an easier, but logical, method. The proofs given below are taken from Chasles' treatise on "Conic Sections" (1865), and are attributed by him to Delbalat.

THEOREM 143.

If two pencils $V\{A, B, \ldots P, Q, \ldots\}$, $W\{A, B, \ldots P, Q, \ldots\}$ are homographic, but not in perspective, the meets $A, B, \ldots P, Q \ldots$ of corresponding rays lie on a conic, which passes through the vertices V, W of the pencils.

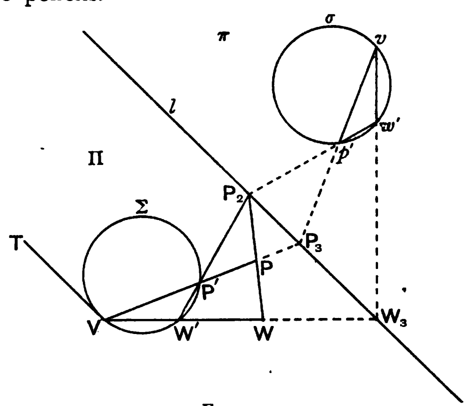

FIG. 125.

Since the pencils are not in perspective, the ray VW is not self-corresponding in the two pencils. Let VT in the V-pencil correspond to WV in the W-pencil.

Draw any circle Σ to touch VT at V and let it cut VW, VA, VP at W', A', P'. Let WP, $W'P'$ meet at P_2.

Now $W\{V, A, \ldots P\} = V\{T, A, \ldots P\}$
$= V\{V, A', \ldots P'\}$
$= W'\{V, A', \ldots P'\}$, by Theorem 57.

But these pencils have a self-corresponding ray $WW'V$.

$\therefore$ they are in perspective.

$\therefore$ the meets $A_2, \ldots P_2$, of WA, $W'A'$; $\ldots WP$, $W'P'$; lie on a line l, (say).

Let VW, VP, meet l at W_3, P_3.

Now rotate the figure out of its plane Π about the line l into a plane π and denote corresponding points by small letters.

Then, since P_2, P_3, W_3 are unaltered by the rotation, corresponding sides of the triangles $vp'w'$, VPW meet on l.

$\therefore$ the triangles $vp'w'$, VPW are in perspective [Theorem 47];

$\therefore$ Vv, Ww', Pp' are concurrent.

Let Vv, Ww' meet at Ω.

Then Pp' passes through Ω.

$\therefore$ similarly, Aa', Bb', Cc', ... pass through Ω.

Now the points a', b', c', ... p', ... lie on the circle σ, *i.e.* the circle obtained by rotating the circle Σ from Π to π.

Therefore the points A, B, C, ... P, ... lie on the projection of the circle σ w.r.t. Ω on the plane Π, which is by definition a conic.

Moreover, since v, w' lie on σ, the conic passes through V, W.

Q.E.D.

If the reader draws Fig. 125 for himself and then folds the paper with the crease along l, he will see more clearly the nature of the projection.

THEOREM 144.

If two ranges $\{A, B, \ldots P, Q, \ldots\}$, $\{A', B', \ldots P', Q' \ldots\}$ are homographic, but not in perspective, the joins AA', BB', ...PP', QQ', ... of corresponding points envelope a conic, touching the two bases α, α' of the ranges.

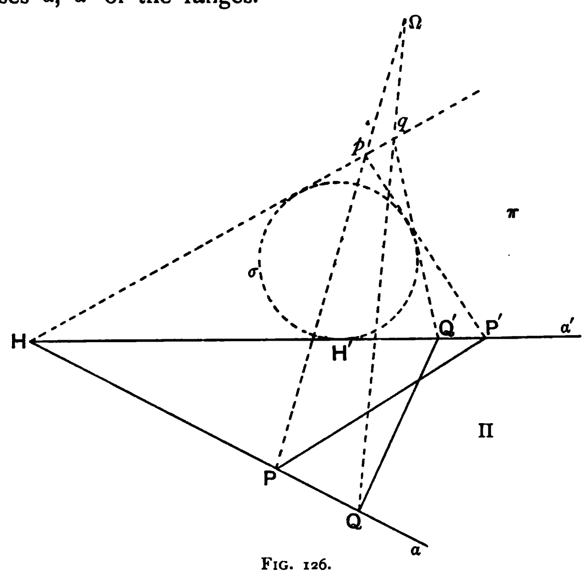

FIG. 126.

Since the ranges are not in perspective, the meet H of α, α' is not a self-corresponding point for the two ranges. Let H' in the α'-range correspond to H in the α-range.

Draw a circle σ in a plane π, different from the plane Π of a, a', to touch a' at H'. Let the tangents to σ from A', B', ... P', ... meet the tangent to σ from H in a, b, ... p, ...

Then $\{H, A, B, \dots P, \dots\} = \{H', A', B', \dots P', \dots\}$

$= \{H, a, b, \dots p, \dots\}$, by Theorem 57.

But these ranges have a self-corresponding point H.

Therefore Aa, Bb, ... Pp, ... are concurrent, at Ω say.

Now consider the projection of the figure in plane π on plane Π w.r.t. Ω.

HP', Hp, aA', bB', ... pP', ... project into HP', HP, AA', BB', ... PP', ...

Also by definition, σ projects into a conic.

Therefore AA', BB', ... PP', ... envelope a conic, which also touches HP, HP', *i.e.* a, a'. Q.E.D.

61. A variable point P lies on a given conic S; Q is a point such that PQ subtends given angles at two fixed points on S; find the locus of Q.

62. P is a variable point on a fixed diameter of a conic; N is the foot of the perpendicular from P to its polar; prove that the locus of N is a rectangular hyperbola.

63. AB is a fixed chord of a given conic S; from a variable point P on the tangent at A, another tangent is drawn, touching S at Q; find the locus of the meet of BP, AQ.

64. If in Theorem 144, the conic is a parabola, prove that the ranges are similar; prove also the converse.

65. P is a variable point on a fixed tangent PT to a parabola; PP_1 is the other tangent from P; PP_2 is drawn, making a given angle in a given sense with PP_1; prove that it envelopes a parabola, touching PT.

66. ABC is a triangle; A is fixed, B moves on a fixed line and $A\hat{B}C$ is of constant size; find the envelope of BC.

67. H, K are the poles of the chords PQ, RS of a conic; prove that H, K, P, Q, R, S lie on a conic.

68. A triangle circumscribes a given conic; two of its sides are fixed in position; find the locus of its circumcentre.

69. A, B, C are three fixed points; a circle is drawn through A, B and another circle through B, C; if their radical axis is fixed, prove that the tangents at A, C meet on a fixed conic.

70. AP, AQ, two variable conjugate lines w.r.t. a given conic, pass through a fixed point A and meet a fixed line at P, Q; if PQR is an equilateral triangle, find the locus of R.

71. HK is a fixed chord of a parabola; a variable line PQ perpendicular to the axis cuts HK at P; the polar of P cuts PQ at Q; find the locus of Q.

72. AB is a given arc of a circle; P_1, P_2 are variable points on the arc AB such that the arc BP_2 is double the arc AP_1; if AP_1, BP_2 meet at P, prove that the locus of P is a hyperbola. Deduce a method of trisecting an angle, with the aid of a conic.

73. A variable chord PQ of a fixed conic passes through a fixed point O; prove that the polar of O w.r.t. the circle on PQ as diameter envelopes a parabola.

74. If a quadrangle is inscribed in a conic, prove that the tangents at its vertices, and one pair of opposite sides touch a conic.

75. A variable line l passes through a fixed point; P is its pole w.r.t. a given conic; prove that the perpendicular from P to l envelopes a parabola. What happens if the fixed point lies on an axis of the conic?

76. The base of a triangle passes through the meet of two common tangents to two given conics, and the ends of the base lie one on each conic. If the sides pass through fixed points, one on each conic, find the locus of the vertex.

77. AB is a given chord of a fixed conic; the tangent at a variable point P on the conic meets the tangent at A in Q; find the envelope of the line through Q, parallel to BP.

78. P is a variable point on a fixed line; PQR is a line, fixed in direction, cutting two fixed lines at Q, R; X is the harmonic conjugate of P w.r.t. Q, R; prove that the locus of X is a hyperbola, having one asymptote parallel to the fixed direction.

79. Through a fixed point O, a variable line is drawn, cutting two fixed lines at P, Q; find the locus of a point, dividing PQ in a constant ratio.

80. A variable line l meets two fixed lines in points which are conjugate w.r.t. a given conic; find the envelope of l.

81. A variable point lies on a fixed line; its polars w.r.t. two given coaxal circles meet at Q; prove that the locus of Q is a conic.

82. The base QR of a variable triangle PQR is a chord of a given conic S; PQ, PR cut S again at fixed points; if QR passes through a fixed point, find the locus of P.

83. A is the pole of a fixed chord BC of a given conic; two variable parallel tangents to the conic cut AB, AC at P, Q respectively; find the envelope of PQ.

84. D is a variable point on the base BC of a given triangle ABC; a parallel through D to AB cuts AC at E; a parallel through E to BC cuts AD at M; prove that the locus of M is a parabola through C, touching AB at A, with its axis parallel to BC.

85. A variable polygon is inscribed in a given conic, and all its sides, save one, pass through fixed points; find the envelope of the remaining side.

86. A straight line passes through a fixed point; prove that the join of its poles w.r.t. two given conics envelopes a conic, inscribed in the common self-conjugate triangle of the given conics.

87. A, B are two fixed points on a conic S; the base QR of a variable triangle PQR is a chord of S and touches a given conic having double contact with S; if PQ, PR pass through A, B respectively, find the locus of P.

88. A, B are fixed points on a conic S; V is a variable point on S, and C is any other fixed point. If $V\{A, B, C, D\}$ is constant, find the envelope of VD.

89. PQR is a variable triangle, inscribed in a fixed circle; PQ is of constant length and PR passes through a fixed point; find the envelope of QR.

90. Prove that a pencil of diameters to a conic is homographic to the pencil of their conjugate diameters.

91. (1) A, B, C, D are four points on a conic, centre O; E, F, G, H are the mid-points of AB, BC, CD, DA; OE', OF', OG', OH' are parallels through O to AB, BC, CD, DA. Prove that

$$O\{E\ F\ G\ H\} = O\{E'\ F'\ G'\ H'\}.$$

[Use Ex. 90.]

(2) **A variable conic passes through four fixed points; prove that the locus of its centre is a conic passing through the diagonal points and the mid-points of the six sides of the quadrangle formed by the four fixed points.** [Use (1).]

92. 123456 is a hexagon inscribed in a conic; prove that the points 1; 2; (13, 24); (23, 41); (15, 26); (25, 16); lie on a conic.

93. A tangent at a variable point P on a parabola meets a fixed tangent at Q; find the locus of a point dividing PQ in a constant ratio.

94. P, P' are conjugate points w.r.t. a given conic; P moves on a fixed line and PP' subtends a constant angle at a given point; find the locus of P'.

95. The bases of two homographic ranges meet at O; P, P' are a pair of corresponding points; find the locus of the mid-point of PP'.

96. P is a variable point on a fixed line; its polar w.r.t. a fixed conic meets another fixed line in Q; find the envelope of PQ.

97. PQR is a variable triangle inscribed in a given conic; if its centroid is a fixed point, prove that the sides of the triangle touch a fixed conic.

In Chapter IV. (page 118), the following theorem [Theorem 76] was proved:

A, *B* are two fixed points. If *P* is a variable point, such that *PA*, *PB* are conjugate lines w.r.t. a fixed conic, then the locus of *P* is a conic, through *A*, *B*.

98. In Theorem 76, prove that the conic-locus cuts the given conic at the point of intersection of the polars of *A*, *B* with the given conic.

99. In Theorem 76, prove that *AB* has the same pole w.r.t. both conics.

100. *a*, *b* are two fixed lines: *P* is a variable point on *a*; *Q* is the point on *b* conjugate to *P* w.r.t. a given conic; prove that *PQ* envelopes a conic having the same polar of *ab* as the given conic.

101. *A*, *B* are the poles of two chords *PQ*, *RS* respectively of a given conic; prove that the six lines *AP*, *AQ*, *PQ*, *BR*, *BS*, *RS* touch a conic.

102. Prove that the conic locus in Theorem 76 breaks up into two straight lines, if AB touches the given conic; that one of these lines is AB, and the other is the join of the points of contact of the other tangents from A, B to the given conic.

THEOREM 145.

The locus of a point from which the tangents to a central conic are at right angles is a circle, concentric with the conic.

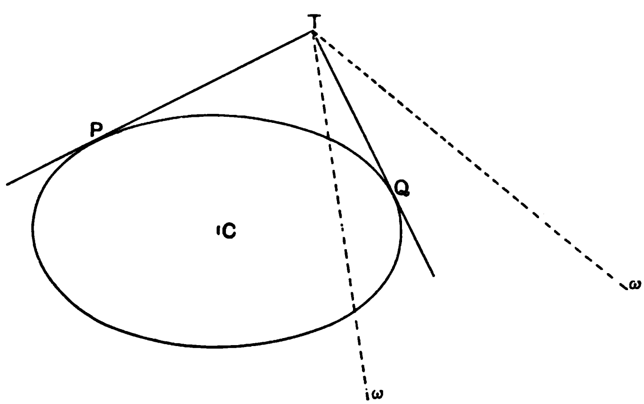

FIG. 127.

Let *TP*, *TQ* be a pair of perpendicular tangents to a conic, centre *C*. Denote the circular points at infinity by ω, ω'.

Then $T\omega$, $T\omega'$ are harmonically conjugate w.r.t. *TP*, *TQ*, since $P\hat{T}Q = 90^\circ$.

$\therefore$ $T\omega$, $T\omega'$ are conjugate lines w.r.t. the conic (Theorem 54).

$\therefore$ by Theorem 76, the locus of T is a conic through ω, ω'.

$\therefore$ the locus of T is a circle.

And by symmetry, the centre of the circle must be at C.

Q.E.D.

Corollary.

If the conic is a parabola, the locus of a point from which the pair of tangents are at right angles is the directrix (and the line at infinity).

The proof is left to the reader. [Use Ex. 102.]

Definition.

The locus of a point, from which the tangents to a central conic are at right angles, is called the **director circle** of the conic.

The property of the director circle was discovered by De Lahire (1640-1718), who wrote on conics, epicycloids, roulettes, conchoids and magic squares. Its analogy with the directrix of the parabola was pointed out by Boscovich (1711-1787); but the name appears to be due to Gaskin.

103. If a, b are the lengths of the semi-axes of a central conic; prove that the radius of the director circle is $\sqrt{a^2+b^2}$.

104. Prove that the director circle of a rectangular hyperbola is a point circle.

105. Prove that the director circle of a conic passes through the (imaginary) points of intersection of the conic with its directrices.

106. A chord PQ of the director circle, centre C, touches the conic; prove that CP, CQ are conjugate diameters.

107. A, A' are a pair of opposite vertices of a quadrilateral touching a conic S at L, M, N, P; if A, A' are projected into the circular points at infinity, prove that the conic through $AA'LMNP$ is projected into the director circle of the projection of S.

108. σ is the director circle of a conic S; prove that the reciprocal of σ w.r.t. S is confocal with S.

109. Prove that the tangent from any point on the director circle of a conic to its auxiliary circle is equal to the minor axis.

110. An ellipse of given size slides between two fixed perpendicular lines; find the locus of its centre.

111. P is a point on the directrix of a conic, focus S; PT is the tangent from P to the director circle; prove that PT = PS.

112. $ABCD$ is a rectangle circumscribing a conic; AB meets a directrix in P; if S is the corresponding focus, prove that $P\hat{S}A = P\hat{B}S$.

113. A variable parallelogram is circumscribed to an ellipse S_1 and its sides are parallel to conjugate diameters of another ellipse S_2; find the locus of its vertices.

114. Two conics are such that an unlimited number of quadrilaterals can be inscribed in one, and circumscribed to the other. If ABCD is any one such quadrilateral, prove that AC, BD meet at a fixed point. [Use projection.]

115. Prove that the focus is a limiting point of the coaxal system having the directrix as radical axis and the director circle as a circle of the system.

116. T is the pole of a chord PQ of a conic; prove that T is a limiting point of the coaxal system formed by the director circle and the circle on PQ as diameter.

117. PQ is a diameter of the director circle of a conic S; prove that the tangents from P, Q to S are parallel or meet in pairs on the director circle.

118. A pair of conjugate diameters of an ellipse cut the director circle at P, Q; prove that PQ touches the ellipse.

THEOREM 146. [APOLLONIUS' THEOREM.]

The feet of the four normals to a conic, centre C, from any point O, lie on a rectangular hyperbola, which passes through C, O and has its asymptotes parallel to the axes of the conic.

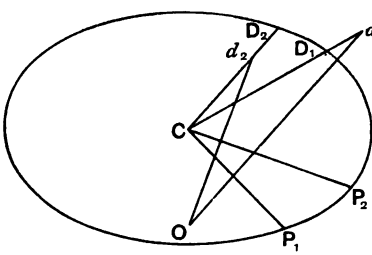

FIG. 128.

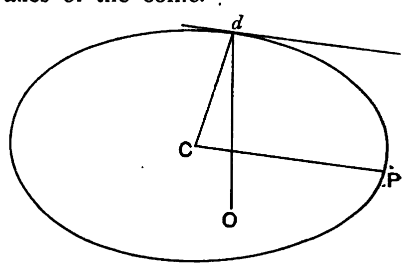

FIG. 129.

Let CP_1, CD_1; CP_2, CD_2; ... be pairs of conjugate diameters of the conic.

Let perpendiculars from O to CP_1, CP_2, ... meet CD_1, CD_2, ... at d_1, d_2, ...

Then

$O\{d_1, d_2, ...\} = C\{P_1, P_2, ...\}$ since the pencils are equiangular.
$\qquad = C\{D_1, D_2, ...\}$ since the diameters are conjugate.
$\qquad = C\{d_1, d_2, ...\}$;

$\therefore$ the points d_1, d_2, ... lie on a conic σ through O, C.

If d is a point of intersection of σ with the given conic, since Cd is conjugate to CP, the tangent at d is parallel to CP and is therefore perpendicular to Od.

$\therefore$ d is the foot of one of the normals from O to the given conic.

$\therefore$ σ meets the given conic at the feet of the normals from O.

Let CA, CB be the axes of the given conic.

Then the perpendicular from O to CA meets the conjugate diameter CB at infinity.

$\therefore$ σ passes through the point at infinity on CB and similarly through the point at infinity on CA.

$\therefore$ σ is a rectangular hyperbola, passing through O, C and having its asymptotes parallel to the axes of the given conic. Q.E.D.

This rectangular hyperbola is called the **hyperbola of Apollonius.** The method of proof is due to Chasles.

119. Prove that four lines can be drawn from a given point to cut a given conic at a given angle (in a determinate sense) and that the four points of intersection, the given point and the centre of the conic lie on a conic.

120. P, Q, R, S are the feet of the four normals from a point to a conic; prove that the tangents at P, Q, R, S touch a parabola, which touches the axes of the conic. [Use reciprocation.]

THEOREM 147.

One and only one conic can be drawn through five given points, no three of which are collinear.

Let A, B, C, D, E be the given points.

Then two homographic non-perspective pencils exist and are determined uniquely by $A\{C, D, E\}$, $B\{C, D, E\}$.

By Theorem 143, the meets C, D, E, ... of corresponding rays lie on a conic through A, B. And only one such conic exists, for any point P on a conic through A, B, C, D, E must by Theorem 57 be a meet of corresponding rays AP, BP of the two pencils; consequently if there were two such conics, any point of either must lie on the other. Q.E.D.

THEOREM 148.

One and only one conic can be drawn to touch five given lines, no three of which are concurrent.

The proof is left to the reader.

THEOREM 149.

The projection of a conic is a conic.

Let A, B be two fixed points on the conic and P a variable point on the conic; then the pencils, $A\{\dots P \dots\}$, $B\{\dots P \dots\}$ are homographic.

$\therefore$ in the projected figure, the pencils $a\{\dots p \dots\}$, $b\{\dots p \dots\}$ are homographic.

$\therefore$ the locus of p is a conic. Q.E.D.

It is beyond the scope of this volume to deal with generalisations of Theorems 143, 144. A concise and interesting account will be found in Chapter XI. of Dr. Filon's treatise on *Projective Geometry*. The two fundamental theorems, there established, are as follows:

THEOREM 150.

If $O\{A, B, \dots P, \dots\}$ is a pencil homographic with the pencil of tangents $\{a', b', \dots p', \dots\}$ to a conic, the locus of the meet of OP, p' is in general a cubic, having a double point at O.

THEOREM 151.

If $\{a, b, \dots p, \dots\}$, $\{a', b', \dots p', \dots\}$ are two homographic pencils of tangents to two different conics, the locus of the meet of p, p' is in general a quartic.

A special case of Theorem 151 has been already dealt with; see page 248. A special case of Theorem 150 is given in Theorem 152.

THEOREM 152.

a is the tangent OA from a fixed point O to a conic σ. If the pencil $O\{A, B, \dots P, \dots\}$ is homographic to the pencil of tangents $\{a, b, \dots p, \dots\}$ to σ, then the locus of the meet of OP, p is a conic.

Draw any tangent z to σ, and let l be any straight line.

Let the required locus cut l at the point E, the meet of OE with the tangent e to σ.

Let a, b, e, ... p, ... meet z at a_1, b_1, e_1, ... p_1, ... and let OA, OB, ... OP, ... meet l at A_1, B_1, ... P_1, ...

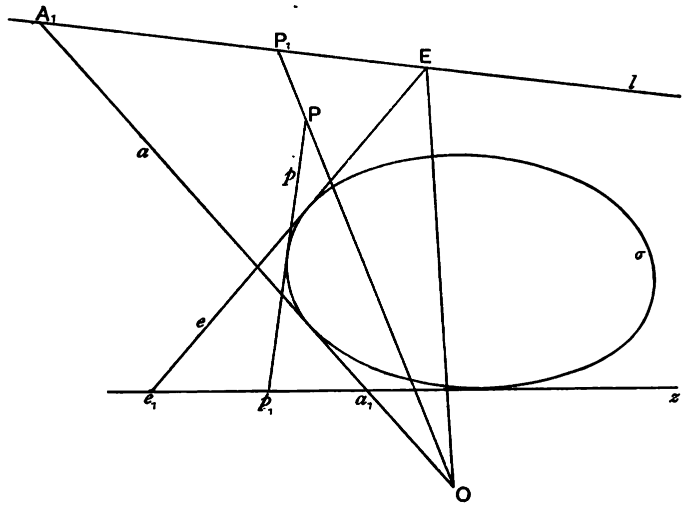

FIG. 130.

$$\text{Then } \{a_1, b_1, e_1, \ldots p_1, \ldots\} = \{a, b, e, \ldots p, \ldots\}$$
$$= O\{A, B, E, \ldots P, \ldots\}, \text{ by hypothesis,}$$
$$= O\{A_1, B_1, E, \ldots P_1, \ldots\};$$

∴ a_1A_1, b_1B_1, e_1E, ... p_1P_1, ... touch a conic S, touching l and z.

Now two conics have four common tangents.

But z, a are two common tangents of S, σ; also Ee_1 is a common tangent of S, σ. Therefore there are two and only two possible positions for Ee_1.

∴ the required locus cuts l at two and only two points.

But l is any straight line whatever.

∴ the required locus is a curve of the second degree, and is therefore a conic. Q.E.D.

121. Write down the dual of Theorem 150.

122. Write down the dual of Theorem 151.

123. Use the method of Theorem 152 to prove Theorem 150.

124. A fixed line l cuts a conic in A; the range of points $\{A, B, C, \ldots P, \ldots\}$ on l is homographic to the range $\{A, B', C', \ldots P', \ldots\}$ on the conic; prove that PP' envelopes a conic.

CHAPTER X.

INVOLUTION RANGES AND PENCILS.

THERE is a special case of the general theory of homographic ranges and pencils, which is of peculiar importance. Its treatment forms the substance of this chapter. The fundamental involution property of the quadrangle was given by Pappus, in rather a complicated form; but the general principles, for six points or six lines in involution, were first enunciated by Desargues. The theory of involution ranges and pencils is worked out in great detail in the *Géométrie Supérieure* of Chasles, to whom the revival of interest in the methods of Desargues is due.

ANALYTICAL TREATMENT.

If two homographic ranges are situated on the same base, to every point ξ of that base, there correspond in general two distinct points, according as ξ is regarded as belonging to the first or second range. Under certain conditions, however, these two points coincide for *all* positions of ξ. These conditions are specified in the following theorem.

(I) If $pxx' + qx + rx' + s = 0$ is the homographic relation connecting two ranges on the same base, referred to the same origin, then $q = r$ is the necessary and sufficient condition that to any point α on the base, there corresponds the same point β, whichever range α belongs to.

(1) To prove the condition is necessary.

If $x = \alpha$, $x' = \beta$ and if $x' = \alpha$, $x = \beta$, for all values of α;

$$\therefore\ p\alpha\beta + q\alpha + r\beta + s = 0,$$

and

$$p\beta\alpha + q\beta + r\alpha + s = 0;$$

$$\therefore \text{ subtracting, } (\alpha - \beta)(q - r) = 0.$$

But $\alpha - \beta \neq 0$, for all values of α, since this requires

$$p\alpha^2 + (q+r)\alpha + s = 0.$$

$$\therefore\ q - r = 0 \text{ or } q = r.$$

(2) To prove the condition is sufficient.

If $q = r$, the relation becomes $pxx' + qx + qx' + s = 0$.

If $x = \alpha$, $x' = -\dfrac{q\alpha + s}{p\alpha + s}$.

If $x' = \alpha$, $x = -\dfrac{q\alpha + s}{p\alpha + s}$.

Therefore the two points, corresponding to the point α on the base, coincide. Q.E.D.

1. If y, z' are the points corresponding to the point ξ on the common base of the ranges, defined by $pxx' + qx + rx' + s = 0$, prove that

$$y - z' = \frac{(q-r)[p\xi^2 + (q+r)\xi + s]}{(p\xi + r)(p\xi + q)}.$$

2. Interpret geometrically the fact that in Ex. 1, $y - z' = 0$ if ξ satisfies the equation $p\xi^2 + (q+r)\xi + s = 0$.

3. If the homographic relation is $xx' - 3x - 3x' + 8 = 0$, find the points corresponding to points whose distances from the origin are 0, 1, 2, 3, 4, ∞.

4. If in Ex. 3, Ω is the point corresponding to $x \to \infty$, find the homographic relation, when the origin is transferred to Ω.

5. If the homographic relation is $xx' - a(x + x') + b = 0$, and if (Ω, ∞) are a pair of corresponding points, find the homographic relation, when the origin is transferred to Ω.

Definition.

If two cobasal ranges are determined by the relation

$$pxx' + q(x + x') + s = 0,$$

referred to the same origin, they are said to be **in involution.**

Suppose then that $\{A, B, C, \ldots\}$, $\{A', B', C', \ldots\}$ are two ranges in involution: since the relation is homographic, the cross-ratio of any four points of one range is equal to that of the corresponding four points of the other range.

But further, if A is now regarded as belonging to the second range, the point corresponding to it in the first range, by (I), is A': and similarly for B, C, ...

It therefore follows that the ranges $(A, A', B, B', C, C', \ldots)$, $(A', A, B', B, C', C, \ldots)$ are homographic: and this is the **characteristic feature of two homographic ranges [A], [A′], in involution.**

It is of the utmost importance that the reader should have a clear idea of the meaning of this property. To every value of x, there corresponds one and only one value of x', given by the homographic relation. Suppose x receives any set of values whatever, α, β, γ, δ, ... ; then we can find uniquely the corresponding values of x'; let these be α', β', γ', δ', ... Since the involution relation, $pxx' + q(x+x') + s = 0$ is symmetrical in x and x', it follows that if x is now given the values α', β', γ', δ', ..., then x' will take the values α, β, γ, δ, ... And therefore the set of values α, α', β, β', γ, γ', ... correspond homographically to the set of values α', α, β', β, γ', γ, ...: and the cross-ratio determined by any four, from one set, is equal to the cross-ratio determined by the corresponding four, from the other set.

(II) A range of points in involution exists and is determined uniquely, if any two pairs of collinear corresponding points are given.

The equation $pxx' + q(x+x') + s = 0$ contains two **independent** constants $\frac{q}{p}, \frac{s}{p}$. Therefore, if two pairs of values of x, x' are given, there are two linear equations from which $\frac{q}{p}, \frac{s}{p}$ can be determined in one and only one way. Q.E.D.

We shall assume at present that, in the involution relation, $p \neq 0$. This passes over a special case, which will be dealt with later (page 267).

Since $p \neq 0$, the involution relation may be written

$$xx' + \frac{q}{p}(x+x') + \frac{s}{p} = 0$$

$$\text{or} \quad xx' + Q(x+x') + S = 0,$$

$$\text{or} \quad (x+Q)(x'+Q) = Q^2 - S.$$

Now change the origin to the point $x = -Q$, and the relation becomes $XX' = Q^2 - S$.

The point, $x = -Q$, has a geometrical interpretation.

Writing the involution relation in the form

$$1 + Q\left(\frac{1}{x'} + \frac{1}{x}\right) + \frac{S}{xx'} = 0$$

we see that when $x' \to \infty$, $\frac{1}{x'} \to 0$ and x is then given by

$$1 + \frac{Q}{x} = 0 \text{ or } x = -Q.$$

The point, $x = -Q$, therefore corresponds to the point at infinity on the base: and will be denoted by O.

Definition.

The point O on the base which corresponds to the point at infinity on the base, in an involution range, is called the **centre of the involution.**

The reader will note that, with the notation of Chapter VIII., for an involution range the points I, J' coincide with the centre O of the involution.

The previous transformation can now be stated as follows: If the centre of an involution range is taken as origin, the involution relation takes the form $XX' = Q^2 - S$.

In geometrical language, we may state it as follows: If O is the centre of an involution range, and if P, P' are a pair of corresponding points, then $OP \, . \, OP'$ is constant.

It was proved in Chapter VIII. that two cobasal homographic ranges have a pair of double points, real or imaginary.

For two ranges in involution, by putting $X = X' = \xi$, we have $\xi^2 = Q^2 - S$ or $\xi = \pm\sqrt{Q^2 - S}$.

Consequently two ranges in involution have a pair of **double points**, real or imaginary, which are equidistant from the centre O of the involution.

The two double points will be denoted by E, F.

Since $OE^2 = OF^2 = Q^2 - S = OP \, . \, OP'$, it follows that every pair of corresponding points are harmonically conjugate w.r.t. the double points.

6. Determine the involution relation, defined by the pairs of points (1, 2); (5, 7). Find the double points; and reduce the relation to its simplest form, by a change of origin.

7. Determine the involution relation, defined by the pairs of points (1, 10); (4, 7). Determine the double points, and a point-pair, whose distance apart is 15.

8. An involution is defined by the relation $xx' - 4x - 7x' + 8 = 0$, referred to two distinct origins for x and x'. Determine the distance between the origins; and reduce the relation to its simplest form.

9. Find the value of a, if the point-pairs (2, 5); (1, 8); (3, a) are in involution.

10. Determine the involution relation, the double points of which are given by $\xi^2 + 2a\xi + b = 0$.

11. X, Y are two fixed points on the base of an involution range; prove that the harmonic conjugates w.r.t. X, Y of the points of the involution range, also form an involution range.

12. a, b are fixed points; p, p' are a pair of variable points on ab such that $ap \,.\, bp' + \lambda \,.\, ap + \mu \,.\, bp' + \nu = 0$; find the condition that p, p' generate involution ranges.

13. Find the condition that the point at infinity is a double point of the involution defined by $pxx' + q(x + x') + s = 0$; and in this case, prove that the finite line joining a variable pair of corresponding points has a fixed mid-point. Where is the second double point?

14. a, a'; b, b'; c, c' are three point-pairs in involution; α, β, γ are the mid-points of aa', bb', cc'; prove that $\dfrac{ab \,.\, ab'}{ac \,.\, ac'} = \dfrac{\alpha\beta}{\alpha\gamma}$. [Take a as origin.]

15. With the notation of Ex. 14, prove that $\dfrac{ab \,.\, ab'}{a'b \,.\, a'b'} = \dfrac{ac \,.\, ac'}{a'c \,.\, a'c'}$.

16. With the notation of Ex. 14, prove that $\dfrac{ab \,.\, ab'}{ac \,.\, ac'} = \dfrac{ab' + a'b}{ac' + a'c}$.

17. E, F are two fixed points; P, P' are a variable pair of points on EF, such that $\dfrac{EP}{FP} + \dfrac{EP'}{FP'} = 0$; prove that P, P' generate ranges in involution.

18. A, B are two fixed points; P, P' are a variable pair of points on AB, such that $\dfrac{AP \,.\, AP'}{BP \,.\, BP'}$ equals a constant λ; prove that P, P' generate ranges in involution; that A, B are a pair of corresponding points; and determine the condition that the double points are real.

19. Prove that the three pairs of points given by $\xi^2 + 2a_1\xi + b_1 = 0$; $\xi^2 + 2a_2\xi + b_2 = 0$; $\xi^2 + 2a_3\xi + b_3 = 0$ are in involution, if

$$a_1(b_2 - b_3) + a_2(b_3 - b_1) + a_3(b_1 - b_2) = 0.$$

[Use the fact that a pair of corresponding points are harmonically conjugate w.r.t. the double points.]

20. Prove that the double points of the involution defined by the pairs of points $\xi^2 + 2a_1\xi + b_1 = 0$; $\xi^2 + 2a_2\xi + b_2 = 0$ are given by

$$(a_1 - a_2)\xi^2 + (b_1 - b_2)\xi + a_2 b_1 - a_1 b_2 = 0.$$

21. (1) The relation connecting two homographic ranges, referred to the same origin, is $pxx' + qx + rx' + s = 0$; prove that there exists a single point A such that if the x-origin is transferred to A, the relation becomes $pXx' + q(X + x') + t = 0$.

(2) Hence prove that if two ranges are homographic, it is possible to superpose uniquely one on the other, in such a way that the two ranges are in involution.

22. Prove that any transversal is cut by a system of coaxal circles in point-pairs, in involution.

[Take the transversal as x-axis and note that any one of the circles can be written in the form $x^2 + y^2 + 2gx + 2fy + c + 2\lambda(lx + my + 1) = 0$, where λ is the only variable, and use Ex. 19.]

THE CASE WHERE $p=0$.

The double points of the involution $pxx'+q(x+x')+s=0$ are given by $p\xi^2+2q\xi+s=0$, by putting $x=x'=\xi$.

This is a quadratic with two roots, real or imaginary.

If $p \to 0$, one root of this quadratic $\to\infty$.

Therefore the involution, $q(x+x')+s=0$ has one of its double points at infinity; or in other words, the centre of the involution is (from the definition) at infinity.

Further, since $\dfrac{x+x'}{2}=-\dfrac{s}{2q}$, if P, P' are any pair of corresponding points, the mid-point of PP' is the point $-\dfrac{s}{2q}$, and is therefore a fixed point, which coincides with the second double point, given by $2q\xi+s=0$, as is geometrically evident.

The involution therefore consists of pairs of points which are equidistant from a fixed point E, which is the other double point.

23. Interpret the condition, $s=0$, in the involution relation

$$pxx'+q(x+x')+s=0.$$

24. The range $\{A_1, B_1, C_1 \dots$ is in involution with the range $\{A_2, B_2, C_2, \dots\}$; the range $\{A_2, B_2, C_2, \dots\}$ is in involution with the range $\{A_3, B_3, C_3, \dots\}$, and $\{A_3, B_3, C_3, \dots\}$ is in involution with $\{A_4, B_4, C_4, \dots\}$; if the point at infinity is a double point for each pair of ranges, prove that the ranges $\{A_1, B_1, C_1, \dots\}$, $\{A_4, B_4, C_4, \dots\}$ are in involution.

25. If, in Ex. 24, the three pairs of ranges have any fixed point as a common double point, prove that the ranges $\{A_1\}$, $\{A_4\}$ are in involution.

26. The range $\{A\}$ is in involution with each of the ranges $\{A'\}$, $\{A''\}$; prove that the ranges $\{A'\}$, $\{A''\}$ are in involution, if and only if the double points of $\{A\}$, $\{A'\}$ are harmonically conjugate w.r.t. the double points of $\{A\}$, $\{A'$.

(III) The three pairs of points given by $a_1\xi^2+2b_1\xi+c_1=0$; $a_2\xi^2+2b_2\xi+c_2=0$; $a_3\xi^2+2b_3\xi+c_3=0$ are in involution,

if and only if
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$$

(1) To prove the condition is necessary.

Let the point-pairs belong to the involution defined by

$$pxx'+q(x+x')+s=0.$$

Now if x, x' are the roots of the equation $a_1\xi^2+2b_1\xi+c_1=0$,

we have $$x+x'=-\frac{2b_1}{a_1};\quad xx'=\frac{c_1}{a_1};$$

$$\left.\begin{array}{lr} \therefore\ p\dfrac{c_1}{a_1}+q\left(-\dfrac{2b_1}{a_1}\right)+s=0 \text{ or } & a_1s-2b_1q+c_1p=0 \\ \text{Similarly} & a_2s-2b_2q+c_2p=0 \\ \text{and} & a_3s-2b_3q+c_3p=0 \end{array}\right\}\ldots\ldots\ldots\ldots(1)$$

$\therefore$ on eliminating $s:q:p$, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0;$$

$\therefore$ the condition is necessary.

(2) If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$, it is possible to find values of p, q, s which satisfy equations (1).

But these equations are the conditions that the given point-pairs are corresponding points in the involution $pxx'+q(x+x')+s=0$.

$\therefore$ the condition is sufficient. Q.E.D.

(IV) The double points of the involution defined by the point-pairs $a_1\xi^2+2b_1\xi+c_1=0$; $a_2\xi^2+2b_2\xi+c_2=0$ are given by

$$\begin{vmatrix} 1 & -x & x^2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$$

Let $\xi=x$ be a double point.

Then the three point-pairs

$$\xi^2-2\xi x+x^2=0$$
$$a_1\xi^2+2b_1\xi+c_1=0$$
$$a_2\xi^2+2b_2\xi+c_2=0$$

are in involution.

$$\therefore \text{ by (III), } \begin{vmatrix} 1 & -x & x^2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$$

Q.E.D.

27. Determine the double points of the involution defined by the point-pairs, $(\sqrt{3}+1,\ \sqrt{3}-1)$; $(\sqrt{-1},\ -\sqrt{-1})$.

28. A, A'; B, B' are two fixed collinear pairs of points; P, P' are a pair of variable points, such that $\{AA'BP\}=\{A'AB'P'\}$; prove analytically that P, P' generate involution ranges.

29. Prove that all involutions defined by the two point-pairs, $\xi^2+a\xi+c=0$, $\xi^2+b\xi+c=0$, for different values of a, b, c, have one common pair of corresponding points.

30. Two involution ranges lie on the same base; prove that there exists one and only one point-pair which belongs to each involution.

31. An involution is defined by the point-pairs, $a_1\xi^2+2b_1\xi+c_1=0$; $a_2\xi^2+2b_2\xi+c_2=0$; prove that any other point-pair of the involution is represented by $a_1\xi^2+2b_1\xi+c_1+\lambda(a_2\xi^2+2b_2\xi+c_2)=0$, for a suitable value of λ.

32. A system of conics pass through four fixed points; prove that any transversal is cut by them in involution ranges [use Ex. 31]: **and deduce that two conics can be drawn to pass through four fixed points and touch a given line.**

33. P, P' are a variable pair of points on a fixed line, and are conjugate w.r.t. a fixed conic; prove that they generate involution ranges. [Take the fixed line as x-axis.] Prove also that the double points are the meets of the line with the conic.

34. Prove that four collinear points define three distinct involutions: and that the double points of one involution are the double points of the involution defined by the two pairs of double points of the other two involutions.

35. A_1, A_2; B_1, B_2; C_1, C_2 and A_1, A_3; B_1, B_3; C_1, C_3 are two cobasal involution ranges; prove that in general A_2, A_3; B_2, B_3; C_2, C_3 are not in involution.

PENCILS IN INVOLUTION.

In Chapter VIII., the similarity between the properties of homographic ranges and pencils was pointed out; of necessity, the same analogy subsists in the case of involution ranges and pencils, as will now be briefly indicated. We shall use the notation already adopted in Chapter VIII.

If two homographic pencils have a common vertex, to every line μ (*i.e.* $y=\mu x$) through the vertex, chosen as origin, there correspond in general two distinct lines, according as μ is regarded as belonging to the first or second pencil. Under certain conditions, however, these two lines coincide, for all values of μ. These conditions are specified in the following theorem.

(V) If $pmm'+qm+rm'+s=0$ is the homographic relation connecting two pencils, with a common vertex, referred to the same axes, then $q=r$ is the necessary and sufficient condition that to

any line α, through the vertex, there corresponds the same line β, whichever pencil α belongs to.

The proof is left to the reader.

[Use the method of (I).]

36. Prove (V).

37. If the homographic relation is $mm' - 4m - 4m' + 15 = 0$, find the lines corresponding to $y=0$, $y=x$, $y=2x$, $y=3x$, $x=0$; determine the equations of the two self-corresponding lines.

Definition.

If two pencils, with a common vertex, are connected by the relation $pmm' + q(m + m') + s = 0$, referred to the same axes, they are said to be **in involution.**

Suppose then that $\{a, b, c, \ldots\}$, $\{a', b', c', \ldots\}$ are two pencils in involution; since the relation is homographic, the cross ratio of any four lines of one pencil is equal to the cross ratio of the corresponding four lines of the other pencil.

But further, if a is now regarded as belonging to the second pencil, the line corresponding to it in the first pencil is a'; and similarly for b, c, ...

Consequently, the pencils

$$\{a, a', b, b', c, c', \ldots\}, \quad \{a', a, b', b, c', c, \ldots\}$$

are homographic; and this is the **characteristic feature of two homographic pencils [a], [a′] in involution.**

(VI) A pencil of lines in involution exists and is determined uniquely, if any two pairs of concurrent corresponding lines are given.

The proof is left to the reader.

[Use the method of (II).]

38. Prove (VI).

DOUBLE LINES OF AN INVOLUTION PENCIL.

It was proved in Chapter VIII. that two homographic pencils, with a common vertex, have a pair of double lines, real or imaginary.

For two pencils in involution, putting $m = m' = \mu$, we see that the double lines are given by $p\mu^2 + 2q\mu + s = 0$

$$\text{or} \quad \mu = \frac{-q \pm \sqrt{q^2 - ps}}{p}.$$

There does not exist any geometrical analogue of the centre of

an involution range. It is possible, however, by a change of axes, to reduce the fundamental relation to a simpler form. Take as axes the two lines bisecting the angles between the two double lines.

Referred to the new axes, the double lines are of the form

$$y = \pm \mu x.$$

Let $Pmm' + Q(m+m') + S = 0$ be the new relation.

Then $\pm\mu$ are the roots of $Pm^2 + 2Qm + S = 0$.

$$\therefore \quad Q = 0;$$

$\therefore$ the fundamental relation becomes $Pmm' + S = 0$.

It should be noted that the substitution $m + \frac{q}{p} = M$; $m' + \frac{q}{p} = M'$ in $pmm' + q(m+m') + s = 0$, which would appear to reduce the relation to the form $MM' = k$, has no direct geometrical significance. The actual substitution is suggested in Ex. 39, 41.

39. Prove that the substitution $m = \dfrac{M+\mu}{1-M\mu}$, $m' = \dfrac{M'+\mu}{1-M'\mu}$ gives the original system referred to new axes, obtained by rotating the old axes through an angle $\tan^{-1}\mu$.

40. Prove analytically that the transformation in Ex. 39 changes an involution pencil into another involution pencil.

41. Prove that the relation $\mathbf{pmm'+q(m+m')+s=0}$ is reduced to the form $\mathbf{mm'=k}$, by rotating the axes through an angle $\mathbf{\tan^{-1}\mu}$, given by $\mathbf{q(\mu^2-1)=(p-s)\mu}$.

Interpret geometrically the equation in μ.

(VII) The double rays of pencils in involution are harmonically conjugate w.r.t. each pair of corresponding rays.

The proof is left to the reader.

[Choose axes so that the relation is $mm' = k^2$; then the double lines are $y = \pm kx$; see also Ex. 45.]

42. Prove (VII), by the first method suggested.

43. Determine the involution defined by the line-pairs, $y=x$, $y=3x$; $y=4x$, $y=7x$. Find the double lines.

44. Determine the involution defined by the line-pairs, $y=x$, $y=8x$; $y=3x$, $y=6x$. Find the double lines.

45. If the line-pair $ax^2+2hxy+by^2=0$ is equivalent to $y=mx$, $y=m'x$; prove that $m+m' = -\dfrac{2h}{b}$, $mm' = \dfrac{a}{b}$. Deduce that the line-pair

$$ax^2+2hxy+by^2=0$$

belongs to the involution defined by $pmm'+q(m+m')+s=0$ if

$$ap+bs=2hq.$$

Hence prove (VII).

46. An involution is defined by $mm' - m - 3m' + 5 = 0$, referred to different axes for m, m'; find the angle between the two x-axes.

47. Reduce the involution, defined by $mm' - m - 5m' + 8 = 0$, referred to different axes, to the form $MM' = k$, by a rotation of axes.

48. Determine the involution defined by $x = 0$, $y = 0$; $x = y\sqrt{3}$, $y = x\sqrt{3}$.

49. Find the line corresponding to $x = 0$ in the involution, defined by $x = 2y$, $x = 6y$; $y = 2x$, $y = 6x$.

50. Determine the involution relation if the double lines are

$$(1)\ x^2 - 8xy + 13y^2 = 0;\quad (2)\ x^2 + y^2 = 0.$$

51. What is the geometrical interpretation of $p = 0$ in the relation $pmm' + q(m + m') + s = 0$?

52. O is a fixed point; OP, OP' are a variable pair of lines at right angles; prove that they generate pencils in involution, and find the double lines.

53. O is a fixed point; OP, OP' are a variable pair of lines inclined to each other at a constant angle α; if $\alpha \neq 90°$, prove that OP, OP' cannot generate pencils in involution.

54. Prove that in every involution there exists one pair of corresponding lines at right angles: and that if there is more than one such pair, then every pair is at right angles.

55. Prove that the three line-pairs,

$$x^2 + 2h_1xy + b_1y^2 = 0;\quad x^2 + 2h_2xy + b_2y^2 = 0;\quad x^2 + h_3xy + b_3y^2 = 0$$

are in involution if $b_1(h_2 - h_3) + b_2(h_3 - h_1) + b_3(h_1 - h_2) = 0$.

56. Prove that the double lines of the involution determined by

$$x^2 + 2h_1xy + b_1y^2 = 0;\quad x^2 + 2h_2xy + b_2y^2 = 0$$

are
$$(h_1 - h_2)x^2 + (b_1 - b_2)xy + (b_1h_2 - b_2h_1)y^2 = 0.$$

(VIII) The three line-pairs $a_1x^2 + 2h_1xy + b_1y^2 = 0$;

$$a_2x^2 + 2h_2xy + b_2y^2 = 0;\quad a_3x^2 + 2h_3xy + b_3y^2 = 0$$

are in involution, if and only if

$$\begin{vmatrix} a_1 & h_1 & b_1 \\ a_2 & h_2 & b_2 \\ a_3 & h_3 & b_3 \end{vmatrix} = 0.$$

The proof is left to the reader.

[Use the method of (III).]

(IX) The double lines of the involution defined by the line-pairs $a_1x^2 + 2h_1xy + b_1y^2 = 0$; $a_2x^2 + 2h_2xy + b_2y^2 = 0$ are

$$\begin{vmatrix} x^2 & -xy & y^2 \\ b_1 & h_1 & a_1 \\ b_2 & h_2 & a_2 \end{vmatrix} = 0.$$

The proof is left to the reader.

[Use the method of (IV).]

(X) In every involution, there is one line-pair at right angles. If there is more than one such line-pair, then every line-pair is at right angles: and the double lines are the isotropic lines through the vertex.

(1) Let the involution relation be $pmm' + q(m+m') + s = 0$.

Let m_1, m_1' be a line-pair of the involution.

From the relation, we have $m_1 m_1' = -1$ if $m_1 + m_1' = \dfrac{p-s}{q}$;

i.e. if m_1, m_1' are the roots of the equation

$$q\mu^2 - (p-s)\mu - q = 0.$$

$\therefore$ there is always one line-pair at right angles. Q.E.D.

(2) Let m_2, m_2' be a second line-pair at right angles.

Then m_1, m_1', m_2, m_2' are four roots of the quadratic,

$$q\mu^2 - (p-s)\mu - q = 0.$$

$$\therefore\ q = 0 \quad \text{and} \quad p - s = 0.$$

$\therefore$ the involution relation is $mm' + 1 = 0$ or $mm' = -1$.

$\therefore$ every line-pair is at right angles.

And the double lines are given by $\mu^2 = -1$ or $\mu = \pm\sqrt{-1}$.

$\therefore$ the double lines are the isotropic lines through the vertex. Q.E.D.

For another method of proof of (X), see Ex. 59.

57. Prove (VIII).

58. Prove (IX).

59. Deduce (X) from (IX).

60. Prove that all involutions defined by the line-pairs,

$$x^2 + 2h_1 xy + by^2 = 0\ ;\ x^2 + 2h_2 xy + by^2 = 0,$$

for different values of h_1, h_2, b have one common pair of corresponding lines.

61. Prove that any line-pair of the involution, defined by

$$\mathbf{a_1x^2 + 2h_1xy + b_1y^2 = 0\ ;\ a_2x^2 + 2h_2xy + b_2y^2 = 0}$$

can be represented by

$$\mathbf{a_1x^2 + 2h_1xy + b_1y^2 + \lambda(a_2x^2 + 2h_2xy + b_2y^2) = 0,}$$

for a suitable value of λ.

Find the values of λ which give the double lines of this involution.

62. If the double lines of an involution are at right angles, prove that the lines bisecting the angles between any pair of corresponding lines are fixed.

63. Find the condition that the isotropic lines are a line-pair of the involution in Ex. 61: and prove that it is the same as the condition that the double lines are at right angles.

64. Through a fixed point, are drawn pairs of conjugate lines w.r.t. a fixed circle: prove that they form involution pencils. [Take the circle in the form $x^2+y^2=a^2$ and the point as (o, b).]

65. By taking four concurrent lines in different pairs, it is possible to form three distinct involutions: prove that the double lines of one are harmonically conjugate to the double lines of each of the others.

66. Prove that the line-pairs $s_1=0$, $s_2=0$; $s_1-\lambda s_2=0$, $s_1-\lambda' s_2=0$; $s_1-\mu s_2=0$, $s_1-\mu' s_2=0$, where $s_r\equiv a_r x+b_r y+c_r$, form an involution, if $\lambda\lambda'=\mu\mu'$.

67. Prove that the double lines of the involution, determined by

$$\phi_1\equiv a_1x^2+2h_1xy+b_1y^2=0\,;\ \phi_2\equiv a_2x^2+2h_2xy+b_2y^2=0$$

are given by
$$\frac{\partial\phi_1}{\partial x}\cdot\frac{\partial\phi_2}{\partial y}=\frac{\partial\phi_2}{\partial x}\cdot\frac{\partial\phi_1}{\partial y}.$$

GEOMETRICAL TREATMENT.

The analytical treatment was opened by defining an involution as a system of point-pairs determined by the homographic relation $pxx'+q(x+x')+s=0$: and it was then shown that this relation was easily reducible to the form $xx'=k$.

We shall take this last relation as the starting-point in our geometrical investigation.

Definition.

O is a fixed point on a fixed line; A, A'; B, B'; C, C'; ... are a system of point-pairs on the line, subject to the condition

$$OA\,.\,OA'=k=OB\,.\,OB'=OC\,.\,OC'=\ldots.$$

Then the system of point-pairs A, A'; B, B'; C, C'; are said to form **an involution**. O is called the **centre** of the involution.

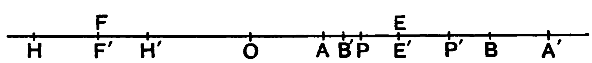

FIG. 131.

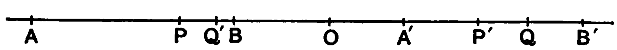

FIG. 132.

It should be noted that, according to this definition, O corresponds to the point at infinity on the base, thus agreeing with the definition on p. 265.

Further, if k is positive (Fig. 131), O is external to the portion of the line joining each point-pair; and the portion of the line joining each point-pair lies wholly inside or wholly outside or wholly includes

the portion of the line joining any other point-pair, *i.e.* they do not overlap.

Whereas, if k is negative (Fig. 132), the portion of the line joining any point-pair contains O, and overlaps the portion of the line joining any other point-pair.

If k is positive, the involution is called **non-overlapping** or **hyperbolic**: and if k is negative, the involution is called **overlapping** or **elliptic**. [The terms "hyperbolic" and "elliptic" refer to the fact that in one case the double points are real, and in the other case imaginary.]

This definition of involution has the advantage of directing attention to one of its principal features, viz. that the unit of an involution is a point-pair, whereas the unit of homographic ranges is a point. It has the disadvantage of obscuring the equally important cross-ratio characteristic.

THEOREM 153.

Given two pairs of points A, A'; B, B' on a straight line l, it is possible to construct a point O on l such that $OA \,.\, OA' = OB \,.\, OB'$.

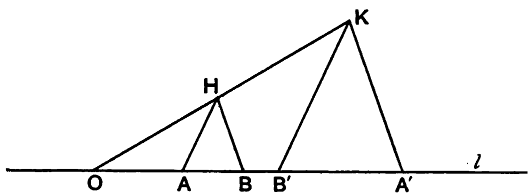

FIG. 133.

Through A, B' draw any two parallel lines to cut any other two parallel lines through B, A' respectively at H, K.

Produce HK to meet l at O.

By parallels, $$\frac{OA}{OB'} = \frac{OH}{OK} = \frac{OB}{OA'}.$$

$$\therefore\ OA \,.\, OA' = OB \,.\, OB'. \qquad \text{Q.E.D.}$$

For another method, see Ex. 68.

N.B. There is nothing in this proof to show that only one position of O exists.

Since $\qquad OA(OA + AA') = (OA + AB)(OA + AB')$,

we have $\qquad OA(AA' - AB - AB') = AB \,.\, AB'$,

which proves that there is only one possible position for O.

Another proof is given on page 277.

68. Prove the following construction for the point O of Theorem 153: H is any point outside l; the circles HAA', HBB' meet again at K; then HK cuts l at the required point O.

69. P, P' is a variable point-pair of a given involution; A is a fixed point outside the base; find the locus of the circumcentre of the triangle APP'.

THEOREM 154.

If A, A'; B, B'; C, C' are three point-pairs in involution, then
$$\{AA'BC\} = \{A'AB'C'\}.$$

FIG. 134.

Let O be the centre of the involution, so that
$$OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC' = k \text{ (say)}.$$

Then $$\{AA'BC\} = \frac{AA' \,.\, BC}{AC \,.\, BA'} = \frac{(OA' - OA)(OC - OB)}{(OC - OA)(OA' - OB)}$$
$$= \frac{\left(\frac{k}{OA} - \frac{k}{OA'}\right)\left(\frac{k}{OC'} - \frac{k}{OB'}\right)}{\left(\frac{k}{OC'} - \frac{k}{OA'}\right)\left(\frac{k}{OA} - \frac{k}{OB'}\right)}$$
$$= \frac{(OA' - OA)(OB' - OC')}{(OA' - OC')(OB' - OA)} = \frac{AA' \,.\, C'B'}{C'A' \,.\, AB'} = \frac{A'A \,.\, B'C'}{A'C' \,.\, B'A}$$
$$= \{A'AB'C'\}.$$ Q.E.D.

Theorem 154 may also be proved as follows:

Let ω be the point at infinity, in a direction perpendicular to the base AB. Then ωA, $\omega A'$, ωB, ωC are the polars of A', A, B', C' w.r.t. the circle, centre O, radius $\sqrt{k}$.

$\therefore$ by Theorem 55, $\{A'AB'C'\} = \omega\{AA'BC\} = \{AA'BC\}$.

Q.E.D.

THEOREM 155.

Given two pairs of points A, A'; B, B' on a base l, there exists one and only one involution, having A, A'; B, B' as pairs of corresponding points.

By Theorem 153, it is possible to find at least one point O, such that $OA \,.\, OA' = OB \,.\, OB'$.

If P is any point on l and if P' is a point such that
$$OP \,.\, OP' = OA \,.\, OA',$$
then the point-pair P, P' generates an involution, having A, A'; B, B' as corresponding points.

But further, the position of the point P', corresponding to P, is uniquely determined, since by Theorem 154,

$$\{A'ABP\} = \{AA'B'P'\}.$$

Therefore there exists one and only one involution, having A, A'; B, B' as point-pairs. Q.E.D.

Corollary.

Given two pairs of points A, ; A' B, B' on a base l, there exists one and only one point O on l, such that $OA \,.\, OA' = OB \,.\, OB'$.

For O corresponds to the point at infinity on l, in the unique involution, defined by A, A'; B, B'.

THEOREM 156.

If A, A'; B, B'; C, C' are three pairs of points on a base l, such that $\{AA'BC\} = \{A'AB'C'\}$, then these three point-pairs form an involution.

If possible, let C'' be the point corresponding to C in the involution, defined by A, A'; B, B'.

By Theorem 154, $\{AA'BC\} = \{A'AB'C''\}$.

But by hypothesis, $\{AA'BC\} = \{A'AB'C'\}$;

$\therefore$ $\{A'AB'C''\} = \{A'AB'C'\}$;

$\therefore$ C'' coincides with C';

$\therefore$ A, A'; B, B'; C, C' form an involution. Q.E.D.

THEOREM 157.

The necessary and sufficient condition that the point-pairs A, A'; B, B'; C, C'; D, D'; ... form an involution is that the ranges $\{A, A', B, B', C, C', D, D', \ldots\}$, $\{A', A, B', B, C', C, D', D, \ldots\}$ should be homographic.

(1) To prove the condition is necessary.

It was proved in Theorem 154, that if the point-pairs are in involution, then ranges of the type $\{AA'BC\}$, $\{A'AB'C'\}$ are equicross; and by a precisely similar method it may be shown that ranges of the type $\{A, B', C', D\}$, $\{A', B, C, D'\}$ are equicross; while ranges of the remaining type $\{AA'B'B\}$, $\{A'ABB'\}$ are equicross (identically). $\therefore$ the condition is necessary.

(2) To prove the condition is sufficient.

Since $\{AA'BP\} = \{A'AB'P'\}$, P, P' are, by Theorem 156, a point-pair of the unique involution defined by A, A'; B, B'; and similarly for any other point-pair.

$\therefore$ the condition is sufficient. Q.E.D.

70. If A, A′; B, B′; C, C′ are three point-pairs in involution, prove that $AB . B'C' . CA' = -A'B' . BC . C'A$.

71. Prove the converse of Ex. 70.

72. With the notation of Ex. 70, prove that $\dfrac{AB . AB'}{AC . AC'} = \dfrac{AB' + A'B}{AC' + A'C}$.

73. If O is the centre of the involution, defined by A, A'; B, B', prove that $\dfrac{OA}{OB} = \dfrac{AB'}{BA'}$.

74. With the notation of Ex. 73, prove that $\dfrac{AB . AB'}{A'B . A'B'} = \dfrac{AO}{A'O}$.

75. Prove that any line is cut in point-pairs in involution by a system of coaxal circles.

76. If P, P′; Q, Q′ are variable point-pairs of a given involution, prove that the radical axis of any two circles, drawn through P, P′ and Q, Q′ respectively, passes through a fixed point.

77. A, A'; B, B'; C, C' are three point-pairs in involution; P is any point outside the base; prove that the circles PAA', PBB', PCC', have a second common point.

78. If A, A′; B, B′; C, C′; ... are point-pairs of an overlapping involution, prove that there exists a real point P, at which AA′, BB′, CC′, ... subtend a right angle.

79. If A, A'; B, B'; C, C' are three point-pairs in involution, prove that $AB = \dfrac{AC}{A'C'} . BC' + \dfrac{CB}{C'B'} . C'A$.

80. With the notation of Ex. 79, prove that $\dfrac{BC . BC'}{BA . BA'} + \dfrac{AC . AC'}{AB . AB'} = 1$.

81. A, A', B, B', C, C' are six collinear points; α, β, γ are mid-points of AA', BB', CC'; P is a variable point on the line; prove that $PA . PA' . \beta\gamma + PB . PB' . \gamma\alpha + PC . PC' . \alpha\beta$ is constant; and that the constant is zero, if A, A'; B, B'; C, C' are three point-pairs in involution.

82. A, A'; B, B'; C, C' are three point-pairs in involution; α, β, γ are the mid-points of AA', BB', CC'; prove that $\dfrac{AB . AB'}{AC . AC'} = \dfrac{\alpha\beta}{\alpha\gamma}$.

83. With the notation of Ex. 82, prove that

$$A\alpha^2 . \beta\gamma + B\beta^2 . \gamma\alpha + C\gamma^2 . \alpha\beta + \alpha\beta . \beta\gamma . \gamma\alpha = 0.$$

84. O is the centre of the involution defined by A, A'; B, B'; if α, β are the mid-points of AA', BB', and if P is any other point on the base, prove that $PB . PB' - PA . PA' = 2PO . \alpha\beta$.

85. If A, A′; B, B′; C, C′ are three point-pairs in involution, prove that the circles, whose diameters are AA′, BB′, CC′, are coaxal.

86. With the notation of Ex. 85, prove that

$$\frac{B'C}{BC'}\cdot AB=\frac{AC}{C'A'}\cdot BB'+\frac{BC}{C'B'}\cdot B'A.$$

87. Prove that each of the sides of a triangle is cut by three circles in point-pairs in involution, if and only if the circles are coaxal.

88. Given five collinear points A, A', B, B', C, construct a point C' such that A, A'; B, B'; C, C' are point-pairs of an involution, without constructing the centre of the involution.

89. If A, A'; B, B'; C, C' are point-pairs in involution, prove that $\frac{BC}{B'C'}:\frac{CA}{C'A'}=\frac{BC'}{B'C}:\frac{C'A}{CA'}$.

[The solution of most of these riders, and a large number of other metrical properties, will be found in Chasles' *Géométrie Supérieure.*]

DOUBLE POINTS.

Definition.

If a point on the base of an involution corresponds to itself, it is called a **double point** of the involution.

A double point of an involution is therefore a point-pair, the two elements of which coincide.

THEOREM 158.

In any involution there exist two double points, equidistant from the centre of the involution, which are real or imaginary, according as the involution is hyperbolic or elliptic.

With the previous notation, any point-pair P, P' is given by

$$OP\,.\,OP'=k.$$

If, then, E is a point on the base, such that $OE\,.\,OE=k$,

$$\text{or } OE^2=k, \text{ or } OE=\pm\sqrt{k},$$

then E is a double point.

$\therefore$ there are two double points E, F given by

$$OE=+\sqrt{k} \text{ and } OF=-\sqrt{k}.$$

E and F are therefore equidistant from O, the centre of the involution: and are real or imaginary according as k is positive or negative, *i.e.* according as the involution is hyperbolic or elliptic.

Q.E.D.

Corollary 1.

Any point-pair of the involution is harmonically conjugate to the double points.

For $$OE^2 = OF^2 = k = OP \,.\, OP'.$$

Corollary 2.

An involution exists and is determined uniquely, when the double points are given.

It should be noted that Corollary 1 is a special case of the general theorem (page 228) that $\{EPFP'\}$ is constant for two cobasal homographic ranges.

THEOREM 159.

(1) If three pairs of points are each harmonically conjugate to two points E, F, then they form an involution, having E, F, as double points.

(2) If three pairs of points are in involution, there exist two points, w.r.t. which each pair is harmonically conjugate.

The proof is left to the reader.

90. Prove Theorem 158, Corollary 2.

91. Prove Theorem 159.

92. If E, F separate harmonically PP′ and QQ′, prove that P, Q; P′, Q′; E, F are three point-pairs in involution.

93. Given two point-pairs P, P'; Q, Q' on a straight line, construct two points X, Y which are harmonically conjugate w.r.t. each point-pair: and state the conditions requisite for X and Y to be real.

94. If $\{XY;\, PP'\}=\{XY;\, QQ'\}=\{XY;\, RR'\}=-1$, prove that
$$\{PP'QR\}=\{PP'R'Q'\}.$$

THEOREM 160.

The necessary and sufficient condition that A, A'; B, B'; C, C' are three point-pairs in involution is

$$AB' \,.\, BC' \,.\, CA' = -A'B \,.\, B'C \,.\, C'A.$$

(1) To prove it is necessary.

If the point-pairs are in involution, $\{AB'BC'\} = \{A'BB'C\}$.

$$\therefore \quad \frac{AB' \,.\, BC'}{AC' \,.\, BB'} = \frac{A'B \,.\, B'C}{A'C \,.\, B'B}.$$

$$\therefore \quad AB' \,.\, BC' \,.\, CA' = -A'B \,.\, B'C \,.\, C'A.$$

Q.E.D.

(2) To prove it is sufficient.

If $$AB'.BC'.CA' = -A'B.B'C.C'A,$$

then $$\frac{AB'.BC'}{AC'.BB'} = \frac{A'B.B'C}{A'C.B'B}.$$

$$\therefore \quad \{AB'BC'\} = \{A'BB'C\}.$$

$\therefore$ A, A'; B, B'; C, C' form an involution. Q.E.D.

THEOREM 161.

If A, A'; B, B'; C, C';... P, P';... are point-pairs of an involution, centre O, then $\dfrac{AP.AP'}{A'P.A'P'}$ is constant for all positions of P, P' and equal to $\dfrac{AO}{A'O}$.

[We have to prove that $AP.A'O.AP' = A'P.AO.A'P'$.]

By hypothesis, $\{APA'O\} = \{A'P'A\infty\}$.

$$\therefore \quad \frac{AP.A'O}{AO.A'P} = \frac{A'P'.A\infty}{A'\infty.AP'} = \frac{A'P'}{AP'}.$$

$$\therefore \quad \frac{AP.AP'}{A'P.A'P'} = \frac{AO}{A'O} = \text{constant}.$$

Q.E.D.

THEOREM 162.

Any transversal is cut by a system of coaxal circles in point-pairs of an involution.

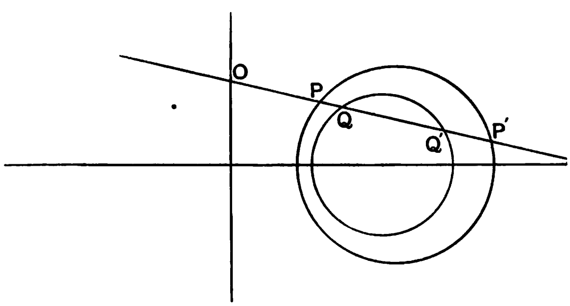

FIG. 135.

Let the transversal cut the radical axis at O, and any one circle at P, P'.

Then $$OP.OP' = \text{constant}.$$

$\therefore$ P, P' generate an involution. Q.E.D.

THEOREM 163.

A, B, C, ... are any system of points on a base l; $A', B', C', \ldots$ are the conjugate points on l, w.r.t. a given conic Σ, of $A, B, C, \ldots$; then the point-pairs A, A'; B, B'; C, C'; ... form an involution on l, having as double points the meets of l with Σ.

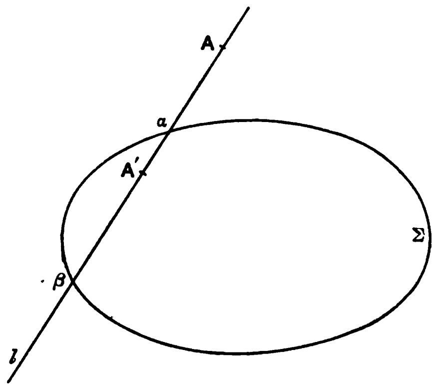

FIG. 136.

Let l meet Σ at the (real or imaginary) points α, β.

Since A, A' are conjugate points w.r.t. Σ, $\{AA' ; \alpha\beta\}$ is harmonic.

Similarly $\{BB' ; \alpha\beta\}$, $\{CC' ; \alpha\beta\}$ etc. are harmonic.

Therefore by Theorem 159 (1), A, A'; B, B'; C, C';.. are point-pairs of an involution, of which α, β are the double points.

Q.E.D.

It is easy to give a proof of Theorem 163, without making use of α, β, see Ex. 95.

95. Prove Theorem 163, without using α, β.

[Let O be the pole of l; prove that OA' is the pole of A and that OA is the pole of A'; deduce from Theorem 55 that

$$\{AA'BB' \ldots\} = \{A'AB'B \ldots\}.]$$

THEOREM 164.

P, P' is a variable point-pair of a given involution; *H* is any fixed point outside the base; then the circle *HPP'* passes through a second fixed point, *i.e.* belongs to a fixed coaxal system.

The proof is left to the reader.

[Let O be the centre of the involution and let OH meet the circle again in K; prove that K is fixed.]

THEOREM 165.

Given two point-pairs A, A'; B, B', to construct the point C' corresponding to a given point C, such that A, A'; B, B'; C, C' form an involution.

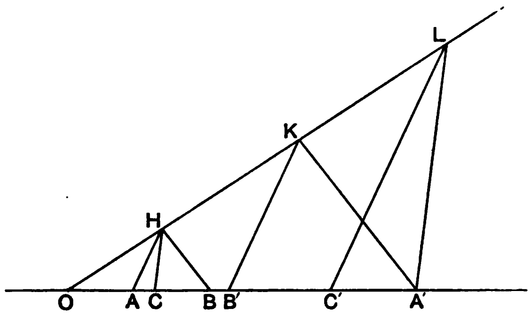

FIG. 137.

First Method.

Take any point H outside the base.

Draw through B', A' lines parallel to AH, BH respectively, to meet at K. Draw $A'L$ parallel to CH to meet HK at L.

Draw LC' parallel to AH to meet the base at C'.

Then C' is the required point.

The proof is left to the reader.

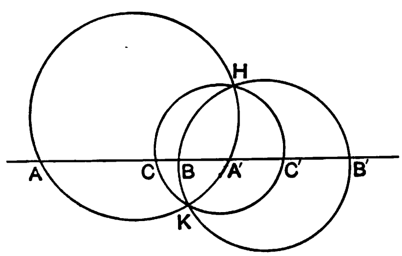

FIG. 138.

Second Method.

Take any point H outside the base.

Draw the circles HAA', HBB' and let them cut again at K.

Draw the circle CHK to cut the base again at C'.

Then by Theorem 164, C' is the required point. Q.E.F.

See also Ex. 230, p. 331.

CONSTRUCTION OF THE DOUBLE POINTS.

If two point-pairs are given, it was shown that the double points are real, if the points of one pair are separated by both or neither of the points of the other pair; but that otherwise the double points are imaginary.

A method for constructing O, given two point-pairs, has already been stated (see page 275). The double points are then given by the meets of the base with a circle centre O, radius $\sqrt{OA \cdot OA'}$, where A, A' is a point-pair; they are in fact the limiting points of the coaxal system formed by the circles whose diameters are AA', BB', CC', ... and are easily constructed.

If the double points are imaginary, this construction gives them as the meets of a real line with an imaginary circle.

We proceed to give a construction, where both line and circle are always real.

THEOREM 166.

O is the centre of an involution defined by the point-pairs, A, A'; B, B'; the lines OA, OA' are rotated in opposite directions, about O, through a right angle into the positions Oa, Oa'; then the circle on aa' as diameter cuts the base at the double points E, F of the involution.

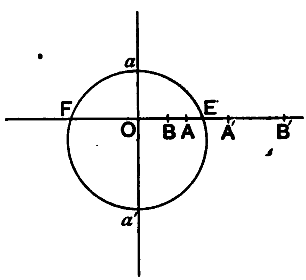

FIG. 139.

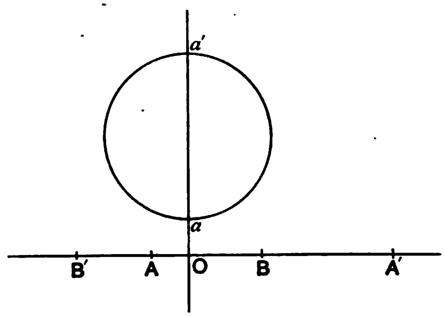

FIG. 140.

Since the base is perpendicular to the diameter aa',

$$FO = OE,$$

also

$$FO \cdot OE = aO \cdot Oa';$$

$$\therefore\ FO^2 = OE^2 = -Oa \cdot Oa'$$

$$= +OA \cdot OA'$$

in either case, Fig. 139 or Fig. 140.

Therefore E, F are the double points. Q.E.D.

[This is a special case of Professor Lodge's construction, p. 226.]

THEOREM 167.

If E, F are the double points of the involution defined by the point-pairs A, A'; B, B', then the point-pairs A, B'; A', B; E, F form an involution; and the point-pairs A, B; A', B'; E, F also form an involution.

By hypothesis, $\{ABEF\} = \{A'B'EF\} = \{B'A'FE\}$.

Therefore by Theorem 156, A, B'; B, A'; E, F are point-pairs in involution.

In precisely the same way, it may be shewn that A, B; A', B'; E, F are also point-pairs in involution. Q.E.D.

Theorem 167 yields **another method for constructing the double points** of an involution defined by two point-pairs A, A'; B, B'.

Take any point H outside the base.

Let the circles HAB', $HA'B$ meet again at K.

Let the circles HAB, $HA'B'$ meet again at L.

Draw the circle HKL.

Then the meets of the circle HKL with the base are the double points.

The proof is left to the reader.

96. Prove Theorem 164.

97. Prove the first method of Theorem 165.

98. Prove the Construction for the double points, on this page.

99. With the usual notation, prove that $\frac{AE^2}{AB'}+\frac{BE^2}{A'B}=AB$.

100. E, F are the double points of a given involution; P, P' are a variable point-pair of the involution; prove that $\frac{EP.FP'}{PP'}$ is constant.

101. With the usual notation, prove that $\frac{EF^2}{EA'}+\frac{AF^2}{EA}=EA$.

102. With the usual notation, prove that $\frac{AB.AB'}{AE^2}=\frac{A'B.A'B'}{A'E^2}$.

103. If one double point of the involution, defined by A, A'; B, B', is at infinity, prove that $AB'+A'B=0$.

104. With the usual notation, prove that $\frac{BE}{B'E}:\frac{AE}{A'E}=A'B:B'A$.

105. With the usual notation, prove that $\frac{AB}{BE}+\frac{AB'}{B'E}=\frac{AA'}{A'E}$.

106. With the usual notation, if α, β are the mid-points of AA', BB', prove that $EF.\alpha\beta=\sqrt{AB.AB'.A'B.A'B'}$.

107. With the notation of Ex. 106, prove that $\frac{\alpha A^2}{\alpha E}-\frac{\beta B^2}{\beta E}=\alpha\beta$.

108. With the notation of Ex. 106, if P is any point on the base, prove that $PE^2 . Fa + PF^2 . aE = PA . PA' . FE$.

109. If $\{AB ; CC'\} = \{BC ; AA'\} = \{CA ; BB'\} = -1$, prove that A, A' ; B, B' ; C, C' form an involution.

110. Prove that two straight lines, divided homographically, can be placed, one on the other, so as to form an involution.

[With the notation of Chapter VIII. place I on J'.]

111. Given three pairs of points on a line, what is the condition for the existence of a pair of points, real or imaginary, harmonically conjugate to each given pair of points? And if they exist, what is the condition for their reality?

112. O is the centre of the involution defined by A, A' ; B, B' ; if B is the mid-point of AA', prove that $\dfrac{OB}{BA} = \dfrac{AB}{BB'}$.

113. If a variable line cuts three fixed circles in involution, prove that it passes through a fixed point, unless the circles are coaxal.

114. Prove that any line through the cross centre of two homographic pencils is cut by the pencils in an involution.

115. PQ is a straight line, which does not cut a conic at real points; the polars of P, Q meet PQ at P', Q' ; prove that PP' and QQ' must overlap.

116. L, L' are the limiting points of the coaxal system defined by two circles; if LL' meets the first circle at A, B and the second at A', B' ; prove that the circles whose diameters are AA', BB', LL' are coaxal.

117. A variable line is drawn through a fixed point A on the radical axis of a coaxal system. Prove that the locus of the double points of the involution so formed is a circle; and that as A varies, these circles generate another coaxal system.

118. E, F are the double points of the involution, defined by A, A' ; B, B'. If BB' is wholly contained in AA', prove that $AB, A'B'$ subtend equal angles at any point on the circle on EF as diameter. What is the corresponding theorem, if AA', BB' are outside each other?

119. A, B, C, A', B', C' are six points in order on a straight line. Construct a point P at which AA', BB', CC' subtend equal angles. [Use Ex. 118.]

120. ABC, $A'B'C'$ are two triangles such that $BC, B'C'$; $CA, C'A'$; $AB, A'B'$ intersect at three collinear points P, Q, R ; AA', BB', CC' meet the line PQR at P', Q', R' ; prove that P, P' ; Q, Q' ; R, R' form an involution.

121. Given a pair of corresponding points A, A' and one double point E of an involution, construct the point corresponding to a given point P.

122. Two given involution ranges are on the same base. Construct the point-pair which belongs to both. Prove that this point-pair is real, if either or both involutions are elliptic; if both are hyperbolic, state the condition that this common point-pair may still be real.

123. Prove that the double points of the involution made on any straight line by a coaxal system are concyclic with the limiting points.

124. A system of conics have ABC as a common self-conjugate triangle; prove that any line through A is cut in involution by the conics.

125. A, A'; B, B'; C, C' form an involution. If $\{AA'; BB'\}$ is harmonic, prove that the harmonic conjugate of C w.r.t. AA' coincides with the harmonic conjugate of C' w.r.t. BB'.

126. P, P' are a variable point-pair of an involution; A is any other fixed point; lines through P, P' perpendicular to AP, AP' meet at P''; prove that the locus of P'' is a straight line.

127. P, P' are a variable point-pair of an involution; E, F are the double points; prove that the circle on PP' as diameter is orthogonal to any circle through E, F.

INVOLUTION PENCILS.

It is easy to see that the theory of involution ranges yields without difficulty an analogous theory of involution pencils. If a system of line-pairs, drawn through a point V, cut any one transversal in point-pairs forming an involution, then every transversal will be cut in involution, by the fundamental cross-ratio property of a pencil of concurrent lines.

Definition.

If a, a'; b, b'; c, c'; ... are a system of line-pairs, drawn through a point V, such that one (and therefore every) transversal is cut by them in an involution range, then the system is said to form an **involution pencil** or to be **in involution**; and V is called the **vertex** of the involution.

The involution pencil is called **elliptic** or **hyperbolic** and **overlapping** or **non-overlapping** according as one (and therefore every) transversal is cut in an overlapping or non-overlapping involution.

THEOREM 168.

Given two pairs of lines a, a'; b, b', which concur at a point V, there exists one and only one involution, having a, a'; b, b' as pairs of corresponding rays.

The proof is left to the reader.

[Consider a section of the pencil.]

THEOREM 169.

(1) If a, a'; b, b'; c, c' are three line-pairs in involution, then $\{aa'bc\} = \{a'ab'c'\}$.

(2) If a, a'; b, b'; c, c' are three pairs of lines, which concur at a common point, and if $\{aa'bc\} = \{a'ab'c'\}$, then a, a'; b, b'; c, c' are the line-pairs of an involution.

The proof is left to the reader.

THEOREM 170.

(1) If a, a'; b, b'; c, c'; d, d'; ... are the line-pairs of an involution, then the pencils $\{aa'bb'cc'dd' \ldots\}$, $\{a'ab'bc'cd'd \ldots\}$ are homographic.

(2) If a, a'; b, b'; c, c'; d, d'; ... are pairs of lines, which concur at a common point, and if the pencils

$$\{aa'bb'cc'dd' \ldots\}, \quad \{a'ab'bc'cd'd \ldots\}$$

are homographic, then a, a'; b, b'; c, c'; d, d'; ... are the line-pairs of an involution.

The proof is left to the reader.

Definition.

If a line through the vertex of an involution pencil corresponds to itself, it is called a **double line** of the involution.

THEOREM 171.

(1) Every involution pencil has two double lines, which are real or imaginary, according as the involution is hyperbolic or elliptic.

(2) An involution pencil exists and is determined uniquely, when the double lines are given.

The proof is left to the reader.

THEOREM 172.

(1) Every line-pair of an involution is harmonically conjugate to the double lines of the involution.

(2) A pair of lines harmonically conjugate to the double lines of an involution are a line-pair of the involution.

(3) If three pairs of lines are each harmonically conjugate to a fourth pair of lines, they form an involution, having the fourth pair as double lines.

The proof is left to the reader.

THEOREM 173.

(1) Every involution pencil has one line-pair at right angles.

(2) If more than one line-pair is at right angles, then every line-pair is at right angles: and the double lines of the pencil are the isotropic lines.

(1) The internal and external bisectors of the angle between the double lines are at right angles and are also a line-pair of the involution by Theorem 172, since they are harmonically conjugate to the double lines.

(2) Let VA, VA'; VB, VB' be two perpendicular line-pairs. By Theorem 168, these two line-pairs determine the involution uniquely. But VA, VA' and VB, VB' are harmonically conjugate to the isotropic lines $V\omega$, $V\omega'$, since $A\hat{V}A' = 90° = B\hat{V}B'$.

$\therefore$ $V\omega$, $V\omega'$ are the double lines of the involution, defined by VA, VA'; VB, VB'.

Further, if VP, VP' is any other line-pair of the pencil,

$$V\{PP';\ \omega\omega'\}$$

is harmonic, by Theorem 172.

$$\therefore\ P\hat{V}P' = 90°.$$

Therefore every other line-pair is at right angles. Q.E.D.

128. If VP, VP' are two variable perpendicular lines through a fixed point V, prove that they generate an involution: and hence deduce another method of proof of Theorem 173 (2).

Definition.

If the line-pairs of an involution are at right angles, the involution is said to be **orthogonal.**

THEOREM 174.

(1) If the isotropic lines through the vertex are a line-pair of an involution, then the double lines are at right angles.

(2) If the double lines of an involution are at right angles, then the isotropic lines through the vertex are a line-pair of the involution.

The proof is left to the reader.

THEOREM 175.

If the angles formed by each of two line-pairs of an involution have the same bisectors, then these bisectors are the double lines of the involution, and the angle formed by any other line-pair has the same bisectors.

The proof is left to the reader.

THEOREM 176.

a, *b*, *c*, ... are any system of lines through a vertex V; a', b', c', ... are the conjugate lines through V, w.r.t. a given conic Σ, of *a*, *b*, *c*, ... ; then the line-pairs a, a'; b, b'; c, c';... form an involution through V, having as double lines the tangents from V to Σ.

The proof is left to the reader.

[Use the dual method of Theorem 163.]

[In the following examples, V denotes the vertex of the involution pencil; VA, VA'; VB, VB'; ... or a, a'; b, b'; ... denote line-pairs of the involution: and VE, VF or e, f denote the double lines.]

129. Prove Theorem 168.

130. Prove Theorem 169.

131. Prove Theorem 170.

132. Prove Theorem 171.

133. Prove Theorem 172.

134. Prove Theorem 174.

135. Prove Theorem 175.

136. Prove Theorem 176.

137. The vertex of an isosceles triangle is fixed, and the base lies on a fixed line; prove that the sides generate an involution.

138. In any involution pencil, prove that there exists in general one and only one line-pair, equally inclined to a given line.

What is the exceptional case?

139. Given two points and their polars w.r.t. a variable conic; prove that the locus of the centre of the conic is a straight line, passing through the meet of the polars. [Use Theorem 176.]

140. If VA, VA' is the perpendicular line-pair of an involution, prove that $\tan AVP \cdot \tan AVP'$ is constant.

141. Prove that $\dfrac{\sin AVP \cdot \sin AVP'}{\sin A'VP \cdot \sin A'VP'}$ is constant.

142. Prove that

$$\sin AVB \cdot \sin B'VC \cdot \sin C'VA' = -\sin A'VB' \cdot \sin BVC' \cdot \sin CVA.$$

143. Prove that two homographic pencils can be superposed so as to form an involution.

144. If two pencils are homographic, prove that there exist a pair of perpendicular rays of one, such that the corresponding rays in the other pencil are also perpendicular.

145. $ABCD$ is a quadrangle inscribed in a conic S; PH, PK are the tangents to S from any point P on the third diagonal; prove that PH, PK; PA, PC; PB, PD are the line-pairs of an involution.

146. $V\{ABCD\}$, $V'\{ABCD\}$ are harmonic pencils; prove that AC is cut by VB, VD; $V'B$, $V'D$ in point-pairs, which generate an involution having A, C as double points.

147. V, V' are conjugate points w.r.t. a conic; PP' is a variable chord through V'; prove that VP, VP' generate an involution. Determine the double lines.

148. Through a point V, lines $V\alpha$, $V\beta$, $V\gamma$ are drawn parallel to the sides BC, CA, AB of a triangle; prove that VA, $V\alpha$; VB, $V\beta$; VC, $V\gamma$ are the line-pairs of an involution.

149. A fixed line cuts a variable line-pair of an involution, vertex V, at P, P'; perpendiculars at P, P' to VP, VP' meet at P''; find the locus of P''.

150. If two pairs of perpendicular lines through a point O are conjugate lines w.r.t. a conic Σ, prove that O must be a focus of Σ.

151. Prove that $\dfrac{\sin^2 AVE}{\sin^2 A'VE}=\dfrac{\sin AVB \,.\, \sin AVB'}{\sin A'VB \,.\, \sin A'VB'}$.

152. Prove that $\sin EVF \,.\, \sin PVP' = 2 \sin EVP \,.\, \sin FVP'$.

153. Prove that conjugate diameters of a conic are line-pairs of an involution. Determine the double lines. Consider also the case where the conic is a circle.

154. PQR is a self-conjugate triangle w.r.t. a system of conics; prove that the tangents from P to any one of the conics generate an involution pencil.

155. E is a meet of the diagonals of a quadrilateral circumscribing a conic S; prove that any line through E is cut in involution by S and the sides of $ABCD$.

156. State and prove the dual of Ex. 155.

157. A is the pole of a fixed line a w.r.t. a conic; tangents from a variable point P on a meet any fixed line through A in P_1, P_2; prove that P_1, P_2 generate an involution range.

PROJECTION.

THEOREM 177.

The projection of a range in involution is a range in involution; and the projection of a pencil in involution is a pencil in involution. Moreover double points and lines project into double points and lines.

The proof is left to the reader.

THEOREM 178.

If A, A'; B, B'; C, C'; ... are the point-pairs of an involution, there exist two positions of a point V, such that the involution pencil formed by VA, VA'; VB, VB'; VC, VC'; ... is orthogonal. And these positions are real, if the involution is elliptic, and are conjugate imaginaries, if the involution is hyperbolic.

The proof is left to the reader.

[Use the method of Theorem 130 page 230, and Theorem 172 (1).]

Theorem 178 may be proved in another way, which is really only superficially different from the method suggested.

Describe two circles on AA', BB' as diameters and let them meet again at V_1, V_2. Then by Theorem 173 (2), V_1, V_2 are the possible positions of V. Further the circles meet at real points, if and only if the involution is overlapping or elliptic.

THEOREM 179.

Any involution pencil can be projected into an orthogonal pencil.

The proof is left to the reader.

[Project VE, VF into isotropic lines; see also Ex. 161.]

158. Prove Theorem 177.

159. Prove Theorem 178, by each method suggested.

160. Prove Theorem 179.

161. Prove that the projection of Theorem 179 may be effected as follows:

Let any transversal cut the pencil at A, A'; B, B'; .... Take, by Theorem 178, a point V in another plane, such that the involution VA, VA'; VB, VB'; ... is orthogonal. Project from V on to a plane, parallel to VAA'.

162. Prove that any involution range can be projected into a system of point-pairs having a common mid-point.

163. A, A'; B, B'; C, C' are three point-pairs on a conic such that their joins to another point on the conic form an involution pencil; prove that their joins to any other point on the conic also form an involution pencil.

164. State and prove the dual of Ex. 163.

165. With the notation of Ex. 163, prove that AA′, BB′, CC′ are concurrent. [Project the conic into a circle having the meet of AA' and BB' as centre.]

166. State and prove the converse of Ex. 165.

167. Prove, by projection, that any transversal is cut in involution by a system of conics through four fixed points.

168. Prove that any transversal is cut in involution by a system of conics, having a common focus and directrix.

169. A variable chord of a given conic subtends a right angle at a fixed point on the conic, prove that it passes through a fixed point. [Use Ex. 165.]

170. X, Y are two fixed points on the base of the involution determined by the point-pairs A, A'; B, B'; CC';.... If A_1, A_1'; B_1, B_1'; C_1, C_1';... are the harmonic conjugates w.r.t. X, Y of A, A'; B, B'; C, C';..., prove that they form an involution.

171. A, A′; B, B′; C, C′ are the pairs of opposite vertices of a quadrilateral; P is any other point; prove that the line-pairs PA, PA′; PB, PB′; PC, PC′ form an involution. [Project $A\hat{P}A'$, $B\hat{P}B'$ into right angles and use the fact that the circles whose diameters are the three diagonals of a quadrilateral are coaxal.]

172. A variable rectangular hyperbola passes through two fixed points A, B; a fixed line l perpendicular to AB cuts the curve at P, P'; prove that P, P' generate an involution on l.

173. Prove that two involution pencils with a common vertex have a single common line-pair. Examine the conditions for its reality.

174. Deduce from Ex. 173, that every conic has in general a single pair of conjugate diameters at right angles, and that this pair is always real.

175. Two conics have a common centre; prove that they have one pair of common conjugate diameters. Examine the condition for reality.

176. A, A', B, B', C, C' are six points on a circle, centre O, such that $\dfrac{\sin\frac{1}{2}C\hat{O}A}{\sin\frac{1}{2}C'\hat{O}A'} \cdot \dfrac{\sin\frac{1}{2}A'\hat{O}B'}{\sin\frac{1}{2}A\hat{O}B} \cdot \dfrac{\sin\frac{1}{2}B\hat{O}C'}{\sin\frac{1}{2}B'\hat{O}C} = -1$; prove that AA', BB', CC' are concurrent. [Use Ex. 165.]

177. Generalise by projection: AB, BC, CD are three tangents to a parabola; BE, CE are parallel to CD, BA; O is any point on BC; OE cuts the parabola at P, Q and AB, CD at R, S; then

$$OP \,.\, OQ = OR \,.\, OS.$$

Write down the dual theorem.

Project the dual theorem so as to obtain a property for a rectangle inscribed in a circle.

178. A, A'; B, B'; C, C'; D, D' are four fixed point-pairs of an involution; H is a variable point on the base; H_1, H_2, H_3, H_4 are the harmonic conjugates of H w.r.t. the four point-pairs; prove that $\{H_1H_2H_3H_4\}$ is constant.

179. P, P' are a variable point-pair of a given involution range; find the locus of the meets of the tangents from P, P' to a given conic. Consider also the special case, when the base touches the conic.

CHAPTER XI.

INVOLUTION PROPERTIES OF THE CONIC.

A, A'; B, B'; C, C'; ... are a system of pairs of points on a conic; V_1, V_2 are any two other points on the conic. By the fundamental cross-ratio property, if the line-pairs V_1A, V_1A'; V_1B, V_1B'; V_1C, V_1C'; ... form an involution pencil, then also the line-pairs V_2A, V_2A'; V_2B, V_2B'; V_2C, V_2C'; ... form an involution pencil. As in Chapter IX., it is therefore unnecessary to specify the particular position of the point V on the conic, when dealing with cross-ratio properties of the pencil $V\{AA'BB'CC' \ldots\}$.

Definition.

A system of point-pairs A, A'; B, B'; ... on a conic is called an **involution range of points on the conic** or an **involution range of the second order**, if their joins to any other point on the conic form an involution pencil.

It follows immediately from the definition that the necessary and sufficient condition that the point-pairs A, A'; B, B'; C, C'; ... on a conic should form an involution is that the ranges

$$\{AA'BB'CC' \ldots\}, \quad \{A'AB'BC'C \ldots\}$$

of points on the conic should be homographic.

THEOREM 180.

An involution range of points on a conic exists and is determined uniquely, when two point-pairs are given.

The proof is left to the reader

[Use Theorem 168 and the method of Theorem 131.]

CHAPTER XI.

INVOLUTION PROPERTIES OF THE CONIC.

a, a'; b, b'; c, c'; ... are a system of pairs of tangents to a conic; v_1, v_2 are any two other tangents to the conic. By the fundamental cross-ratio property, if the point-pairs v_1a, v_1a'; v_1b, v_1b'; v_1c, v_1c'; ... form an involution range, then also the point-pairs v_2a, v_2a'; v_2b, v_2b'; v_2c, v_2c'; ... form an involution range. As in Chapter IX., it is therefore unnecessary to specify the particular position of the tangent v to the conic, when dealing with cross-ratio properties of the range $v\{aa'bb'cc' \ldots\}$.

Definition.

A system of tangent-pairs a, a'; b, b'; ... to a conic is called an **involution pencil of tangents to the conic** or an **involution pencil of the second order,** if their meets with any other tangent to the conic form an involution range.

It follows immediately from the definition that the necessary and sufficient condition that the tangent-pairs a, a'; b, b'; c, c'; ... to a conic should form an involution is that the pencils

$$\{aa'bb'cc' \ldots\}, \quad \{a'ab'bc'c \ldots\}$$

of tangents to the conic should be homographic.

THEOREM 181.

An involution pencil of tangents to a conic exists and is determined uniquely, when two tangent pairs are given.

The proof is left to the reader.

THEOREM 182.

Given an involution range of points on a conic, there exist two and only two points on the conic (real or conjugate imaginaries), each of which is a coincident point-pair of the involution.

The proof is left to the reader.

[Use Theorem 171 and the method of Theorem 133.]

Definition.

If AA'; BB'; CC'; ... are point-pairs on a conic, forming an involution, and if E, F are the two points, each of which is a coincident point-pair of the involution, then E, F are called the **double points** of the involution of points on the conic.

THEOREM 184.

If A, A'; B, B'; C, C'; ... are point-pairs of an involution range of points on a conic, and if E, F are the double points; then the lines AA', BB', CC', ... meet at a point, viz. the pole O of EF w.r.t. the conic.

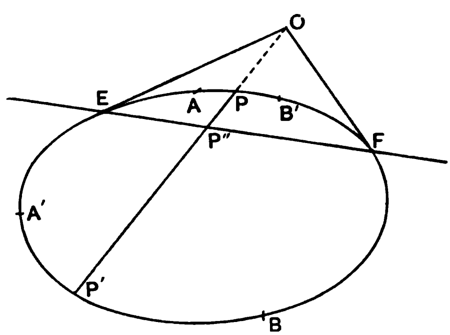

FIG. 141.

By Theorem 172, $E\{PP'; EF\}$ is harmonic;

$\therefore$ $E\{PP'; OF\}$ is harmonic.

Similarly $F\{PP'; OE\}$ is harmonic.

But these pencils have a common corresponding ray EF;

$\therefore$ the meets of EP, FP; EP', FP'; EO, FO are collinear.

$\therefore$ PP' passes through O. Q.E.D.

If E, F are imaginary, the theorem is still true in virtue of the Principle of Continuity.

THEOREM 183.

Given an involution pencil of tangents to a conic, there exist two and only two tangents to the conic (real or conjugate imaginaries), each of which is a coincident tangent-pair of the involution.

The proof is left to the reader.

Definition.

If a, a'; b, b'; c, c'; ... are tangent-pairs to a conic, forming an involution, and if e, f are the two tangents, each of which is a coincident tangent-pair of the involution, then e, f are called the **double lines** of the involution of tangents to the conic.

THEOREM 185.

If a, a'; b, b'; c, c'; ... are tangent-pairs of an involution pencil of tangents to a conic, and if e, f are the double lines; then the points aa', bb', cc', ... lie on a line, viz. the polar o of ef w.r.t. the conic.

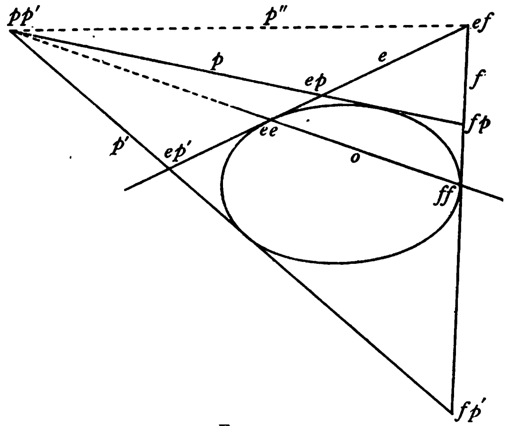

FIG. 142.

By Theorem 158 Cor., $e\{pp'; ef\}$ is harmonic;

$\therefore$ $e\{pp'; of\}$ is harmonic.

Similarly $f\{pp'; oe\}$ is harmonic.

But these ranges have a common corresponding point ef;

$\therefore$ the joins of ep, fp; ep', fp'; eo, fo are concurrent;

$\therefore$ pp' lies on o. Q.E.D.

If e, f are imaginary, the theorem is still true in virtue of the Principle of Continuity.

THEOREM 186.

If A, A'; B, B'; C, C'; ... are pairs of points on a conic, such that the lines AA', BB', CC', ... pass through a point O, then the point-pairs A, A'; B, B'; ... form an involution of points on the conic, having as double points the meets E, F of the conic with the polar of O.

Let P'' be the meet of EF, PP' (see Fig. 141).

Then $E\{EF;\ PP'\} = E\{OP'';\ PP'\} = \{OP'';\ PP'\} = -1$;

$\therefore$ $E\{EF;\ PP'\}$ is harmonic;

$\therefore$ by Theorem 172 (3), EP, EP' generate an involution having EE, EF as double lines;

$\therefore$ P, P' generate an involution of points on the conic, having E, F as double points. Q.E.D.

If E, F are imaginary, the theorem is still true in virtue of the Principle of Continuity.

For riders, see Exx. 1–35, page 308.

THEOREM 188 [DESARGUES' THEOREM].

If a system of conics pass through four fixed points—three of these conics being pairs of lines—then the meets of any line with the conics form point-pairs in involution.

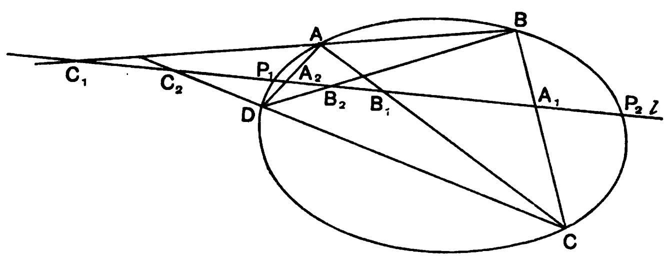

FIG. 143.

A, B, C, D are the fixed points and l is the given line.

Let the meets of l with BC, AD; CA, BD; AB, CD be A_1, A_2; B_1, B_2; C_1, C_2, and let the meets of l with any conic of the system be P_1, P_2.

Then $C\{P_1P_2AB\} = D\{P_1P_2AB\}$;

$\therefore$ $\{P_1P_2B_1A_1\} = \{P_1P_2A_2B_2\} = \{P_2P_1B_2A_2\}$;

$\therefore$ A_1, A_2; B_1, B_2; P_1, P_2 are three point-pairs in involution.

THEOREM 187.

If a, a'; b, b'; c, c'; ... are pairs of tangents to a conic, such that the points aa', bb', cc', ... lie on a line o, then the tangent-pairs a, a'; b, b'; ... form an involution of tangents to the conic, having as double lines the tangents e, f to the conic from the pole of o.

Let p'' be the join of ef, pp' (see Fig. 142).

Then $e\{ef; pp'\} = e\{op''; pp'\} = \{op''; pp'\} = -1$;

$\therefore$ $e\{ef; pp'\}$ is harmonic;

$\therefore$ by Theorem 159 (1), ep, ep' generate an involution, having ee, ef as double points;

$\therefore$ p, p' generate an involution of tangents to the conic, having e, f as double lines. Q.E.D.

If e, f are imaginary, the theorem is still true in virtue of the Principle of Continuity.

THEOREM 189 [STURM'S THEOREM].

If a system of conics touch four fixed lines—three of these conics being pairs of points—then the tangents from any point to the conics form line-pairs in involution.

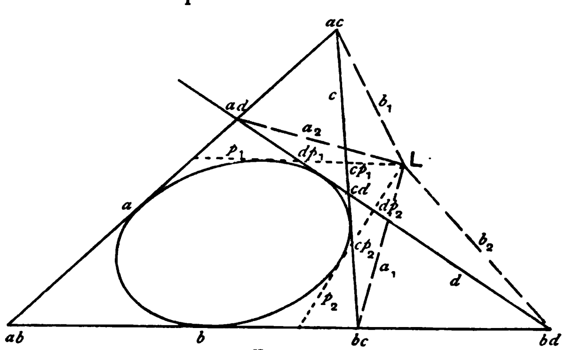

FIG. 144.

a, b, c, d are the fixed lines and L is the given point.

Let the joins of L with bc, ad; ca, bd; ab, cd be a_1, a_2; b_1, b_2; c_1, c_2, and let the tangents from L to any conic of the system be p_1, p_2.

Then $c\{p_1p_2ab\} = d\{p_1p_2ab\}$;

$\therefore$ $\{p_1p_2b_1a_1\} = \{p_1p_2a_2b_2\} = \{p_2p_1b_2a_2\}$;

$\therefore$ a_1, a_2; b_1, b_2; p_1, p_2 are three line-pairs in involution.

Similarly $A\{P_1P_2BC\} = D\{P_1P_2BC\}$;

$$\therefore \{P_1P_2C_1B_1\} = \{P_1P_2B_2C_2\} = \{P_2P_1C_2B_2\};$$

$\therefore$ C_1, C_2; B_1, B_2; P_1, P_2 are three point-pairs in involution. But an involution is defined uniquely by B_1, B_2; P_1, P_2;

$\therefore$ A_1, A_2; B_1, B_2; C_1, C_2; P_1, P_2 form an involution.

Similarly, if Q_1, Q_2 are the meets of l with any other conic of the system, Q_1, Q_2 is also a point-pair of the involution.

Q.E.D.

Corollary.

If X_1, X_2 is any point-pair of the involution on l, a conic can be drawn to pass through the six points A, B, C, D, X_1, X_2.

Other methods of proof are indicated in Exx. 49, 50, and Chapter X., Ex. 32.

Definition.

A system of conics, which pass through four fixed points, is called a **pencil** of conics.

The following properties are immediate consequences of Desargues' Theorem.

(1) If a line l touches one of a pencil of conics, then the point of contact of l is the double point of the involution formed by the point-pairs of the meets of l with the conics.

(2) Two and only two conics can be drawn to pass through four points and to touch a given line.

In particular two and only two parabolas can be drawn through four points.

(3) If PQR is the diagonal point triangle of the quadrangle $ABCD$, the meets of any line through P with the pencil of conics through A, B, C, D form an involution, having P as one double point.

A number of special cases arise, by making two or more of the points A, B, C, D coincide.

For riders on Theorem 188, see Ex. 36-87.

Similarly, $a\{p_1p_2bc\} = d\{p_1p_2bc\}$;

$\therefore \{p_1p_2c_1b_1\} = \{p_1p_2b_2c_2\} = \{p_2p_1c_2b_2\}$;

$\therefore c_1, c_2; b_1, b_2; p_1, p_2$ are three line-pairs in involution.

But an involution is defined uniquely by $b_1, b_2; p_1, p_2$;

$\therefore a_1, a_2; b_1, b_2; c_1, c_2; p_1, p_2$ form an involution.

Similarly, if q_1, q_2 are the tangents from L to any other conic of the system, q_1, q_2 is also a line-pair of the involution.

Q.E.D.

Corollary.

If x_1, x_2 is any line-pair of the involution through L, a conic can be drawn to touch the six lines a, b, c, d, x_1, x_2.

Other methods of proof are indicated in Exx. 98, 99.

Definition.

A system of conics, which touch four fixed lines, is called a **range** of conics.

The following properties are immediate consequences of Sturm's Theorem.

(1) If a point L lies on one of a range of conics, the tangent at L is a double line of the involution formed by the line-pairs of the tangents from L to the conics.

(2) Two and only two conics can be drawn to touch four lines and to pass through a given point.

(3) If pqr is the diagonal line triangle of the quadrilateral $abcd$, the tangents from any point on p to the range of conics touching a, b, c, d, form an involution, having p as one double line.

A number of special cases arise, by making two or more of the lines a, b, c, d coincide.

For riders on Theorem 189, see Ex. 88-115.

THEOREM 190 [LAMÉ'S THEOREM].

(1) If two points P, Q are conjugate w.r.t. each of two conics S_1, S_2, then they are conjugate w.r.t. every conic passing through the four common points of S_1, S_2 (*i.e.* w.r.t. every conic of the pencil determined by S_1, S_2).

(2) The polars of a given point P w.r.t. a pencil of conics are concurrent.

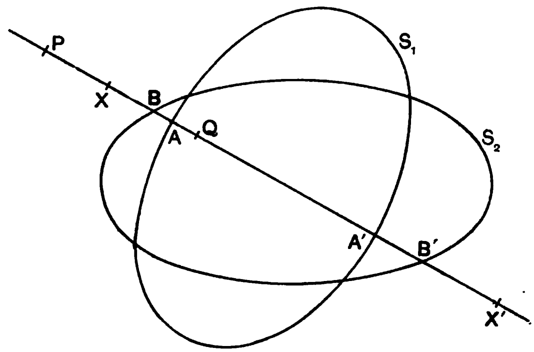

FIG. 145.

(1) Let A, A'; B, B' be the meets of PQ with S_1, S_2.

Then since P, Q are conjugate points,

$$\{AA'; PQ\} \text{ and } \{BB'; PQ\}$$

are harmonic.

Therefore P, Q are the double points of the involution, defined by A, A'; B, B'.

Let X, X' be the meets of PQ with any conic S of the pencil, determined by S_1, S_2.

Then by Theorem 188, A, A'; B, B'; X, X' are in involution;

$$\therefore \{XX'; PQ\} \text{ is harmonic;}$$

$$\therefore P, Q \text{ are conjugate w.r.t. } S.$$ Q.E.D.

(2) Since P, Q are conjugate w.r.t. S, the polar of P w.r.t. S passes through Q.

Therefore the polar of P w.r.t. any conic of the pencil passes through Q.

Therefore the polars of P are concurrent. Q.E.D.

THEOREM 191.

(1) If two lines p, q are conjugate w.r.t. each of two conics s_1, s_2, then they are conjugate w.r.t. every conic touching the four common tangents of s_1, s_2 (*i.e.* w.r.t. every conic of the range determined by s_1, s_2).

(2) The poles of a given line p w.r.t. a range of conics are collinear.

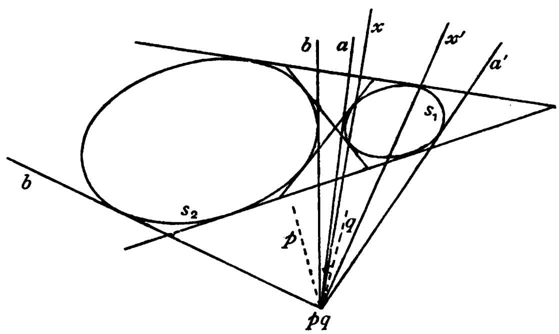

FIG. 146.

(1) Let a, a'; b, b' be the tangents from pq to s_1, s_2.

Then since p, q are conjugate lines,

$$\{aa'; pq\} \text{ and } \{bb'; pq\}$$

are harmonic.

Therefore p, q are the double lines of the involution, defined by a, a'; b, b'.

Let x, x' be the tangents from pq to any conic s of the range, determined by s_1, s_2.

Then by Theorem 189, a, a'; b, b'; x, x' are in involution;

$\therefore$ $\{xx'; pq\}$ is harmonic;

$\therefore$ p, q are conjugate w.r.t. s. Q.E.D.

(2) Since p, q are conjugate w.r.t. s, the pole of p w.r.t. s lies on q.

Therefore the pole of p w.r.t. any conic of the range lies on q.

Therefore the poles of p are collinear. Q.E.D.

THEOREM 192.

(1) P is a variable point on a fixed line l; P' is the point which is conjugate to P w.r.t. two (and therefore all) of a pencil of conics through four fixed points A, B, C, D; then the locus of P' is a conic σ which passes through the poles of l w.r.t. the conics of the pencil.

(2) The conic σ passes through the following eleven points: If l cuts AB at H and if H' is the harmonic conjugate of H w.r.t. A, B, then σ passes through H' and the corresponding five points on the other five sides of the quadrangle $ABCD$; σ also passes through the three vertices of the diagonal point triangle of $ABCD$ and through the two double points of the involution formed by the meets of l with the pencil of conics.

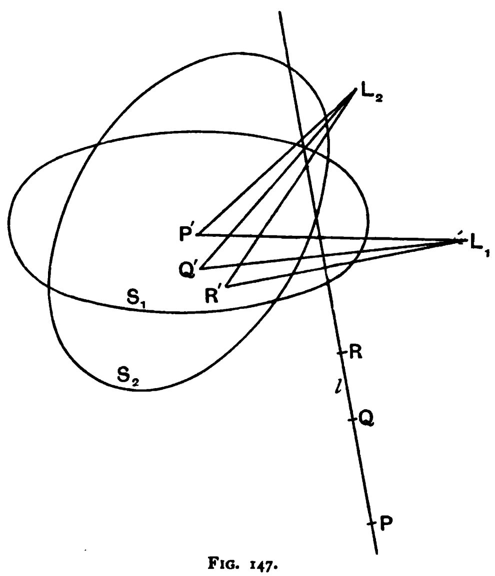

FIG. 147.

(1) Let L_1, L_2, ... be the poles of l w.r.t. the conics S_1, S_2, ... of the pencil.

P, Q, R, ... are a system of points on l; and P', Q', R', ... are their conjugates w.r.t. two and $\therefore$ all the conics of the pencil.

Therefore $P'L_1$, $Q'L_1$, $R'L_1$, ... and $P'L_2$, $Q'L_2$, $R'L_2$, ... are the polars of P, Q, R, ... w.r.t. S_1, S_2;

$$\therefore\ L_1\{P', Q', R', \ldots\} = \{P, Q, R, \ldots\} = L_2\{P', Q', R', \ldots\}.$$

THEOREM 193.

(1) p is a variable line through a fixed point L; p' is a line which is conjugate to p w.r.t. two (and therefore all) of a range of conics touching four fixed lines a, b, c, d; then the envelope of p' is a conic σ which touches the polars of L w.r.t. the conics of the range.

(2) The conic σ touches the following eleven lines: If h is the join of L and ab, and if h' is the harmonic conjugate of h w.r.t. a, b, then σ touches h' and the corresponding five lines through the other five vertices of the quadrilateral $abcd$; σ also touches the three sides of the diagonal line triangle of $abcd$ and the two double lines of the involution formed by the tangents from L to the range of conics.

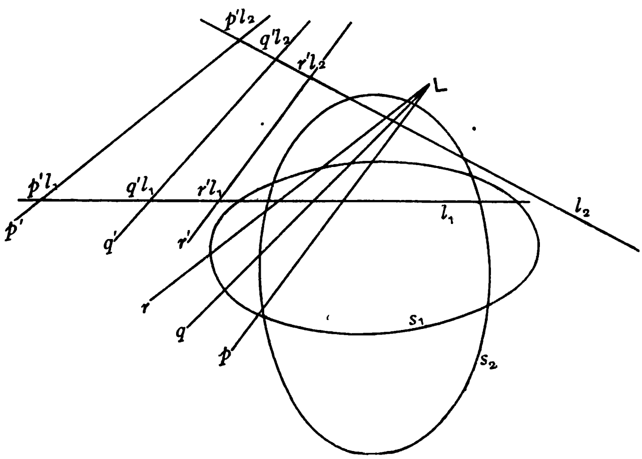

FIG. 149.

(1) Let l_1, l_2, ... be the polars of L w.r.t. the conics s_1, s_2, ... of the range.

p, q, r, ... are a system of lines through L; and p', q', r', ... are their conjugates w.r.t. two and $\therefore$ all the conics of the range.

Therefore $p'l_1$, $q'l_1$, $r'l_1$, ... and $p'l_2$, $q'l_2$, $r'l_2$, ... are the poles of p, q, r, ... w.r.t. s_1, s_2.

$$\therefore\ l_1\{p', q', r', \ldots\} = \{p, q, r, \ldots\} = l_2\{p', q', r', \ldots\}.$$

Therefore the locus of P' is a conic σ through L_1, L_2, and similarly through the other poles L_3, L_4, ... of l w.r.t. the other conics of the pencil. Q.E.D.

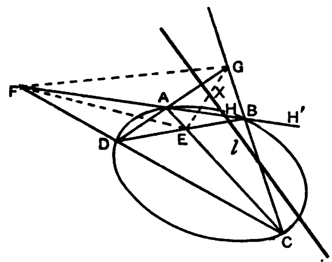

FIG. 148.

(2) Since $\{AB; HH'\}$ is harmonic, H, H' are conjugate points w.r.t. every conic of the pencil.

But H lies on l; $\therefore$ σ passes through H'.

Similarly σ passes through the corresponding five points on AC, AD, BC, BD, CD.

Let EFG be the diagonal point triangle of $ABCD$, and let X be the meet of l and EG.

Then, since EG is the polar of F, $\therefore$ F, X are conjugate points w.r.t. every conic of the pencil.

But X lies on l; $\therefore$ σ passes through F.

Similarly σ passes through E, G.

Lastly, if λ, μ are the double points of the involution on l, λ, μ are conjugate points w.r.t. each conic of the pencil;

$\therefore$ σ passes through λ, μ. Q.E.D.

Definition.

The conic-locus σ of Theorem 192 is called the **eleven-point conic** of the pencil of conics through A, B, C, D corresponding to the line l.

[This name is due to Dr. Taylor.]

THEOREM 194.

The locus of the poles of a given line l w.r.t. a pencil of conics is the eleven-point conic corresponding to l.

This is merely another way of stating Theorem 192.

Therefore the envelope of p' is a conic σ touching l_1, l_2 and similarly the other polars l_3, l_4, ... of L w.r.t. the other conics of the range. Q.E.D.

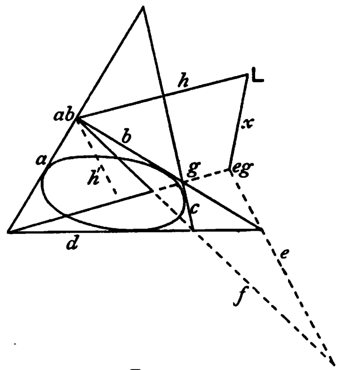

FIG. 150.

(2) Since $\{ab;\ hh'\}$ is harmonic, h, h' are conjugate lines w.r.t. every conic of the range.

But h passes through L; $\therefore$ σ touches h'.

Similarly σ touches the corresponding five lines through ac, ad, bc, bd, cd.

Let efg be the diagonal line triangle of $abcd$, and let x be the join of L and eg.

Then, since eg is the pole of f, $\therefore f$, x are conjugate lines w.r.t. every conic of the range.

But x passes through L; $\therefore$ σ touches f.

Similarly σ touches e, g.

Lastly, if λ, μ are the double lines of the involution through L, λ, μ are conjugate lines w.r.t. each conic of the range;

$\therefore$ σ touches λ, μ. Q.E.D.

Definition.

The conic-envelope σ of Theorem 193 is called the **eleven-line conic** of the range of conics touching a, b, c, d corresponding to the point L.

[This name is due to Dr. Taylor.]

THEOREM 195.

The envelope of the polars of a given point L w.r.t. a range of conics is the eleven-line conic corresponding to L.

This is merely another way of stating Theorem 193.

1. Two chords PQ, RS of a conic meet at H; MN is the tangent at a point M of the conic; prove that MP, MQ; MR, MS; MH, MN form an involution.

2. T is the pole of a fixed chord PQ of a conic; a fixed line through T cuts the conic at H, K; a variable line through H cuts PQ at X and the conic at X'; prove that KX, KX' generate involution pencils.

3. The vertex A of the triangle ABC is a fixed point on a given conic, and the mid-point and direction of BC are fixed. If AB, AC cut the conic again at B', C', prove that $B'C'$ passes through a fixed point.

4. O is a fixed point; p, p' are a variable line-pair of a given involution. P, P' are the feet of the perpendiculars from O to p, p'; prove that PP' passes through a fixed point.

5. A variable triangle is inscribed in a given conic; two of its sides pass through fixed points; prove that its vertices trace out homographic ranges of points on the conic.

6. A variable quadrilateral circumscribes a given conic; three of its vertices lie on fixed lines; prove that its sides generate homographic pencils of tangents to the conic.

7. Prove that the inverse of an involution range of points on a circle is an involution range, either on a circle or on a straight line.

8. What is the reciprocal of an involution range of points on a conic?

9. E is a fixed point on a fixed diameter DD' of a conic; through E is drawn a variable chord PP' of the conic; $D'P$, $D'P'$ cut the tangent at D in Q, Q'; prove that $DQ \cdot DQ'$ is constant.

10. If A, A′; B, B′; C, C′; ... are point-pairs of an involution range on a conic, prove that the point at which AA′, BB′, CC′, ... concur is the pole w.r.t. the conic of the cross-axis of the homographic ranges {A, B, C, ...}, {A′, B′, C′, ...} of points on the conic.

11. State and prove the dual of Ex. 10.

12. A is a fixed point on a given tangent AC to a conic S; P, Q are variable points on AC such that AP, AQ subtend equal angles at a fixed point B; prove that the other tangents from P, Q to S meet on a fixed line.

13. O is a fixed point on a conic S; PQ is a variable chord of S, such that OP, OQ are equally inclined to a fixed line; prove that PQ passes through a fixed point.

14. A variable chord PQ of a conic subtends a right angle at a fixed point on the conic; prove that PQ passes through a fixed point.

15. AB is a common tangent of two conics S_1, S_2; from a variable point T on a fixed line, tangents are drawn to S_1, and meet AB at P, Q; find the locus of the meet of the other tangents from P, Q to S_2.

16. A variable circle passes through two fixed points and cuts a fixed tangent to a conic S at P, Q; find the locus of the meet of the other tangents from P, Q to S.

17. A variable circle cuts two fixed circles orthogonally; prove that the points of intersection generate involution ranges on the fixed circles.

18. A is a common point of two conics S_1, S_2; PQ is a variable chord of S_1 passing through a fixed point; AP, AQ meet S_2 at P', Q'; prove that $P'Q'$ also passes through a fixed point.

19. AB is a fixed diameter of a conic S; C is a fixed point on the tangent at B; a variable line through C cuts S at P, Q; BP, BQ meet the tangent at A in P', Q'; prove that the mid-point of $P'Q'$ is fixed.

20. The tangent AE at A to a conic is parallel to the base BC of the fixed triangle BCD inscribed in the conic: PQ is a variable chord parallel to CD; BP, BQ meet AE at P', Q'; if CD meets AE at E, prove that $EP' . EQ'$ is constant.

21. A fixed circle S cuts a variable circle of a given coaxal system at P, P'; prove that P, P' generate an involution range on S.

22. A is a fixed point on a conic S; a variable pair of parallel tangents to S meet a fixed tangent at P, Q; AP, AQ meet S at P', Q'; find the envelope of $P'Q'$.

23. a, b are two common tangents of the conics S_1, S_2; the tangent at a variable point P of S_1 meets a at Q, the tangent from Q to S_2 meets b at R, the tangent from R to S_1 touches S_1 at T; find the envelope of PT.

24. The circle S is cut orthogonally by each of three circles S_1, S_2, S_3; prove that it is cut by them in an involution, if and only if the circles are coaxal.

25. Tangents are drawn from a variable point on a fixed line to a given conic and meet a fixed tangent to the conic at P, Q; prove that P, Q generate involution ranges.

26. BC is the tangent at B to a conic S; the tangents to S from two points A_1, A_2 meet BC in P_1, Q_1 and P_2, Q_2 respectively; if A_1A_2 meets BC at C, prove that B, C; P_1, Q_1; P_2, Q_2 form an involution.

27. From a variable point on a fixed line, tangents p, q are drawn to a parabola. From a fixed point O, lines OP, OQ are drawn parallel to p, q; prove that OP, OQ generate an involution pencil.

28. Two chords PQ, RS of a conic meet on the chord AB; if O is the pole of AB, prove that the line-pairs AP, AQ; AR, AS; AO, AB form an involution.

29. A, B, C, D, P are five points on a conic; O is any other point; OA, OB cut the conic again at A', B'; prove that

$$O\{ACBD\}=P\{ACBD\}\times P\{A'CB'D\}.$$

30. A fixed circle passes through the centre of a given conic; a variable pair of conjugate diameters of the conic meet the circle at P, Q; prove that PQ passes through a fixed point.

31. PP' is a variable chord of a fixed conic; O is any fixed point. If OP, OP' are a line-pair of a given involution pencil, find the envelope of PP'.

32. A, B, C, P, Q, R are six points on a conic; if

$$R\{PABC\}=R\{QCBA\},$$

prove that PQ meets AC on the tangent at B.

33. T is the pole of a fixed chord PQ of a conic; a variable line cuts TP, TQ at H, K, and PQ at a fixed point R; prove that the other tangents from H, K to the conic meet on a fixed line.

34. OP, OP' is a variable line-pair of a given involution pencil; the corresponding rays of a homographic pencil, with a different vertex, meet OP, OP' at P, P'; prove that PP' passes through a fixed point.

35. OA, OB are two fixed tangents to a conic; two variable parallel tangents meet OA, OB at P, Q respectively; find the envelope of PQ.

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36. E, F are the double points of the involution determined on a straight line by the sides of a quadrangle inscribed in a circle S; prove that the circle on EF as diameter is orthogonal to S.

37. Deduce from Desargues' Theorem that the intercept of any tangent to a hyperbola between the asymptotes is bisected at the point of contact.

38. Two conics touch the same line at P, Q respectively and cut at A, B, C, D; if R is the mid-point of PQ, prove that the conic through A, B, C, D, R has one asymptote parallel to PQ.

39. Two conics cut at A, B, C, D; a straight line touches them at P, Q, and cuts AC, BD at H, K; prove that {PQ; HK} is harmonic.

40. Deduce from Desargues' Theorem a property, by making A, B coincide.

41. O is the mid-point of a chord AB of a conic; C, D are points on AB equidistant from O; two lines CPQ, DRS cut the conic at P, Q and R, S; if PR, QS meet AB at X, Y, prove that $CX=DY$.

42. Deduce from Desargues' Theorem a property, by making A, B, C coincide.

43. P is any point on a chord AB of a conic; a line through P cuts the conic in C, D and the tangents at A, B to the conic in Q, R; prove that $PC.PR.QD=PD.PQ.CR$.

44. H is the mid-point of a chord AB of a conic; PQ, RS are two chords through H; PR, QS meet AB at X, Y; prove that AX=BY.

45. A pair of common chords of two conics S_1, S_2 meet at T; the tangent TP to S_1 cuts S_2 at H, K; prove that $\{TP; HK\}$ is harmonic.

46. A line drawn through a point P on a hyperbola, parallel to an asymptote, cuts two pairs of opposite sides of an inscribed quadrangle in H, H'; K, K'; prove that $PH.PH'=PK.PK'$.

47. The three pairs of opposite sides of a quadrangle inscribed in a hyperbola meet an asymptote in P, P'; Q, Q'; R, R'; prove that $PQ=P'Q'$ and $QR=Q'R'$.

48. A diameter of a parabola meets a chord PQ in H, the tangents at P, Q in M, N and the curve in O; prove that $OH^2=OM.ON$.

49. With the notation of Fig. 143, deduce from Pappus' Theorem that $\dfrac{P_1A_1.P_2B_1}{P_1B_1.P_2A_1}=\dfrac{P_2A_2.P_1B_2}{P_2B_2.P_1A_2}$ and hence prove Desargues' Theorem.

50. Prove Desargues' Theorem, by projecting, in Fig. 143, A, B into the circular points at infinity.

51. PQ is a chord of a conic bisecting another chord AB at O; the tangents at P, Q meet AB in S, T; prove that $AS=BT$.

52. A, B, C, I, J are five points on a conic; IJ, BC meet the tangent at A in S, T; U is the harmonic conjugate of A w.r.t. S, T; prove that A, U are the double points of the involution intercepted on AT by the pencil of conics through $BCIJ$.

What does this theorem become, if I, J are the circular points at infinity?

53. P, Q, R, S are four points on a conic; a line through O, the meet of PS, QR, cuts the conic at A, A' and PQ, RS at B, B'; prove that $\dfrac{1}{OA}+\dfrac{1}{OA'}=\dfrac{1}{OB}+\dfrac{1}{OB'}$.

54. PQ, RS are parallel chords of a conic; another chord HK cuts PQ at O; HR, KS meet PQ at L, M; prove that $OP.OQ=OL.OM$.

55. P, Q, R, S, T are five points on a conic; PQ, RS, PR, QS cut the tangent at T in H, K, M, N; prove that $\dfrac{1}{TH}+\dfrac{1}{TK}=\dfrac{1}{TM}+\dfrac{1}{TN}$.

56. From a fixed point, lines are drawn parallel to the sides of a quadrangle; prove that they form an involution.

57. Show how to draw through a given point H a line cutting the sides of a quadrangle in an involution, having H as centre.

58. Through four given points, show how to draw a conic to intercept on a given line a chord of given length.

59. A variable conic passes through four fixed points A, B, C, D and cuts a fixed conic through A, B in P, Q; prove that PQ passes through a fixed point.

60. A system of conics pass through four fixed points A, B, C, D; any line through A cuts the conics at $P_1, P_2, P_3, \ldots$; if $AT_1, AT_2, AT_3, \ldots$ are the tangents at A to the conics, prove that

$$\{P_1, P_2, P_3, \ldots\} = A\{T_1, T_2, T_3, \ldots\}.$$

61. A variable conic passes through four fixed points A, B, C, D, and cuts two fixed lines AP, AQ at P, Q; prove that the locus of the meet of CP, BQ is a straight line.

62. The circle of curvature at a point P of a conic cuts the conic again at Q; prove that PQ and the tangent at P divide harmonically the other common tangent of the circle and the conic.

63. The tangents at the points P, P′ on a hyperbola meet an asymptote at Q, Q′; prove that PP′ bisects QQ′.

64. A tangent at a point P on a conic meets the auxiliary circle at Q, R and the major axis at T; prove that $\{PT; QR\}$ is harmonic. If N is the foot of the ordinate from P to the major axis, prove that TN bisects $Q\hat{N}R$.

65. A variable chord PQ of a conic passes through a fixed point A; B is another fixed point; BP, BQ meet the conic again at R, S; prove that RS passes through a fixed point.

66. ABC is a given triangle inscribed in a conic; O is a fixed point on the conic; a variable line through O cuts the conic again at P and BC, CA, AB at Q, R, S; prove that $\{PQRS\}$ is constant.

67. From a fixed point O, lines OA', OB', OC' are drawn parallel to the sides BC, CA, AB of a triangle; prove that OA, OA'; OB, OB'; OC, OC' form an involution pencil.

68. E is a point of intersection of two diagonals of a quadrilateral $ABCD$ circumscribing a conic; PQ is a chord passing through E; prove that the six points A, B, C, D, P, Q lie on a conic.

69. A, B, C, D, E are five fixed points on a conic S; the tangent at E to S meets a variable conic through A, B, C, D at P, P'; if Q is the mid-point of PP', prove that $\dfrac{EP \cdot EP'}{EQ}$ is constant.

70. PCP', DCD' are conjugate diameters of an ellipse; the straight lines joining D, D' to a variable point on the ellipse meet the tangent at P in X, Y; prove that $\dfrac{1}{PX} \sim \dfrac{1}{PY} = \dfrac{1}{CD}$.

71. ABC is a fixed triangle inscribed in a conic; P is a variable point on the curve; BP, CP meet the tangent at A in Q, R; prove that $\dfrac{1}{AQ} - \dfrac{1}{AR}$ is constant.

72. Two parabolas touch at P and cut at Q, R; prove that PQ, PR are harmonically conjugate to the diameters through P of the two curves.

73. PP' is a diameter of a conic, centre O; AB, CD are two chords cutting PP' at X, X'; if $PX=P'X'$, prove that any conic through A, B, C, D cuts PP' at points equidistant from O.

74. C is the pole of a chord AB of a conic; DE is another chord of the conic; CD cuts AB, AE, BE at G, H, K; prove that

$$\frac{CD^2}{GD^2}=\frac{CH.CK}{GH.GK}.$$

75. A system of conics circumscribe a given triangle and have a common pair of conjugate points; prove that they pass through a fourth fixed point.

76. Two conics S_1, S_2 have double contact with each other; a chord P_1Q_1 of S_1 cuts S_2 at P_2, Q_2; if P_1Q_1 is parallel to the chord of contact, prove that $P_1P_2=Q_1Q_2$.

Deduce a special case by taking the chord of contact as the line at infinity.

77. A transversal cuts the sides BC, CA, AB of a triangle at P, Q, R; three other points P′, Q′, R′ are taken, such that P, P′; Q, Q′; R, R′ form an involution; prove that AP′, BQ′, CR′ are concurrent.

78. Two conics have double contact at A, B; a line touches one of them at H and cuts the other at P, Q; if it meets AB at K, prove that $\{PQ; HK\}$ is harmonic.

79. O is a point on a chord PQ of a conic; R is a point on PQ such that $OR^2=OP.OQ$. From any point on the polar of R, tangents are drawn to the conic, cutting PQ at H, K; prove that $OH.OK=RO^2$.

80. The sides BC, CA, AB of a triangle touch a conic at D, E, F; PT is the tangent at any point P of the conic; prove that

$$P\{TABC\}=P\{TDEF\}.$$

81. The sides BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ of the triangles ABC, $A'B'C'$ meet in three collinear points P, Q, R; if AA', BB', CC' cut PQ at P', Q', R', prove that P, P'; Q, Q'; R, R' form an involution.

82. l is the line of intersection of the planes of two triangles ABC, $A'B'C'$; BC, CA, AB, $B'C'$, $C'A'$, $A'B'$ meet l at a, b, c, a', b', c'. If Aa', Bb', Cc' are concurrent, prove that $A'a$, $B'b$, $C'c$ are also concurrent.

83. P, Q, R, S are four fixed points on a conic; a variable line through P cuts the conic again at P' and QR, RS, SQ at S', Q', R'; prove that $\{P'Q'R'S'\}$ is constant.

84. Two chords AB, CD of a conic are conjugate w.r.t. the conic; any chord AP meets BC, CD, DB at Q, R, S; prove that $\{PR; QS\}$ is harmonic.

85. A, A' are one pair of opposite vertices of a quadrilateral $ABA'B'$ circumscribing a conic; if P, Q are the points of contact of the conic with AB', $A'B'$, prove that a conic can be drawn through P, Q to touch BA, BA' at A, A' respectively.

86. ABC is a fixed triangle inscribed in a conic; a variable chord PQ meets BC, CA, AB at a, b, c; if $\{Pabc\}$ is constant, prove that Q is a fixed point.

87. Three equidistant lines, parallel to an asymptote of a conic, meet the curve at the vertices of a triangle; prove that any line parallel to the other asymptote is divided harmonically by the curve and the sides of this triangle.

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88. A tangent at a point P to a conic meets two other tangents TQ, TR at Q, R; A is the pole of a chord BC of the conic; prove that AB, AC; AP, AT; AQ, AR form an involution pencil.

89. TP_1, TQ_1; TP_2, TQ_2 are the tangents from a point T to two conics S_1, S_2; if TP_1, TP_2 are conjugate lines w.r.t. a conic S_3, touching the four common tangents of S_1, S_2, prove that TQ_1, TQ_2 are also conjugate lines w.r.t. S_3.

90. O is one of the common points of two conics, and OT_1, OT_2 are the tangents at O to the conics; if A, A' is one pair of opposite vertices of the quadrilateral formed by their common tangents, prove that $O\{AA'; T_1T_2\}$ is harmonic.

91. Deduce a property from Theorem 189 by making a, b, c coincide.

92. Deduce a property from Theorem 189 by making a, b and also c, d coincide.

93. Through a given point, show how to draw a pair of lines which will divide each of the joins of opposite vertices of a complete quadrilateral harmonically.

94. $ABCD$ is a quadrilateral circumscribing a conic; a tangent at any point P on the conic cuts CD at Q; AP, BP cut CD at L, M; prove that $\{QMLC\}=\{QDCL\}$.

95. $ABCD$, $A'B'C'D'$ are two quadrilaterals circumscribing a conic; if AA', BB', CC' concur at a point O, prove that DD' also passes through O.

96. $ABCD$ is a quadrangle inscribed in a conic S; PH, PK are the tangents to S from any point P on the third diagonal; prove that a conic can be inscribed in $ABCD$ to touch PH, PK.

97. AA'; BB'; CC' are the pairs of opposite vertices of a quadrilateral; $\{AA'; PQ\}$ and $\{BB'; RS\}$ are harmonic ranges; prove that PR and QS cut CC' harmonically.

98. $ABCD$ is a quadrilateral circumscribing a circle, centre O; AD, BC meet at E; AB, CD meet at F; prove that the angles between the lines OA, OE and OC, OF are equal; and deduce that OA, OC; OB, OD; OE, OF form an involution of which the isotropic lines through O are one line-pair: hence deduce, by projection, Theorem 189.

99. With the notation of Fig. 144, project L into the orthocentre of the projection of the triangle acd and the line b to infinity: hence deduce Theorem 189 from the theorem that the orthocentre of a triangle circumscribing a parabola lies on the directrix.

100. A system of conics touch three fixed lines and have a given pair of lines as conjugate lines; prove that they touch a fourth fixed line.

101. A, A'; B, B'; C, C' are the opposite vertices of a quadrilateral; T is any point on a conic through $AA'BB'$; CC' meets this conic at P, Q; prove that TP, TQ are the tangents to the two conics which can be inscribed in the quadrilateral and pass through T.

102. Prove that the feet of the perpendiculars from the six vertices of a quadrilateral to any straight line form an involution range.

103. A, A'; B, B' are two pairs of opposite vertices of the quadrilateral formed by the four common tangents of two conics which cut orthogonally at a point P; prove that $A\hat{P}B = A'\hat{P}B'$.

104. HK is a chord of a conic inscribed in the quadrilateral $PQRS$; PR, QS meet at O; OH, OK meet the curve again at H', K'; prove that a conic can be inscribed in $PQRS$ to touch $HK', H'K$.

105. Given a point O, a conic through O and four fixed lines l_1, l_2, l_3, l_4; construct, with the use of a ruler only, the tangents from a given point on l_1 to the two conics which pass through O and touch l_1, l_2, l_3, l_4.

106. Four circles have one common point; prove that their radical axes form an involution pencil.

107. TP, TQ are the tangents from a point T to a conic, foci S, H; prove that the angles STP, HTQ are equal or supplementary.

108. Prove that confocal conics intersect at right angles.

109. O, O' are the centres of similitudes of two circles S_1, S_2; PH_1, PK_1 and PH_2, PK_2 are the tangents from any point P to S_1, S_2; prove that PO, PO'; PH_1, PK_1; PH_2, PK_2 form an involution pencil.

110. ABC is a fixed triangle circumscribing a given conic S; TP is a tangent from a variable point T to S; if $T\{ABCP\}$ is constant, prove that the locus of T is a straight line.

111. ABC is a triangle circumscribing a conic; the polar of A meets BC at D; a tangent at any point Q of the conic meets the other tangent from D in T; prove that $T\{BC; QA\}$ is harmonic.

112. A variable conic, inscribed in the fixed triangle ABC, touches BC at a fixed point; H, K are fixed points on BC; prove that the locus of the meet of the other tangents from H, K to the conic is a straight line through A.

113. P is any point on a conic inscribed in the quadrilateral $ABCD$; the tangent at P, PB, PD meet AC in T, Q, R; prove that

$$\frac{AT^2}{CT^2}=\frac{AQ.AR}{CQ.CR}.$$

114. The poles of two fixed lines w.r.t. a variable conic are fixed points; prove that the locus of the centre of the conic is a straight line.

115. A variable conic is inscribed in the quadrilateral $ABA'B'$; E, F are fixed points on AB, AB'; prove that the other tangents from E, F to the conic meet on a fixed line through A'.

THEOREM 196 [FRÉGIER'S THEOREM].

If a variable chord PP' of a conic subtends a right angle at a fixed point V on the conic, then it passes through a fixed point F situated on the normal at V.

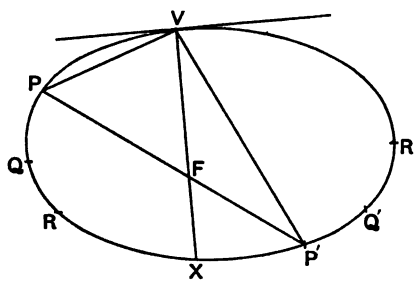

FIG. 151.

Let QQ', RR', SS', ... be other positions of the chord.

Since $P\hat{V}P'=90°=Q\hat{V}Q'=R\hat{V}R'=...$, the line-pairs VP, VP'; VQ, VQ'; VR, VR'; ... form an involution;

$\therefore$ P, P'; Q, Q'; R, R'; ... form an involution range of points on the conic;

$\therefore$ PP', QQ', RR', ... are concurrent by Theorem 184.

But one position of PP' is the normal VX at V, since $V\hat{V}X=90°$;

$\therefore$ PP' passes through a fixed point F on the normal at V.

Q.E.D.

Definition.

If a variable chord of a conic subtends a right angle at a fixed point V on the conic, the fixed point F through which the chord passes, is called the **Frégier point** of the point V.

116. If V is a point on a rectangular hyperbola, prove that the Frégier point of V is at infinity.

117. QR is a chord of a rectangular hyperbola parallel to the normal at a point P on the curve, prove that $Q\hat{P}R = 90°$.

118. Prove Frégier's Theorem by reciprocating w.r.t. V.

119. P, Q are variable points on a fixed tangent to a conic, and subtend a right angle at a fixed point; find the locus of the meet of the other tangents from P, Q to the conic.

120. F is the Frégier point of a point V on a parabola; prove that VF is bisected by the axis of the parabola. [Consider, in Fig. 151, the chord PP' when P' is at infinity.]

121. F is the Frégier point of a point V on a central conic. If ACA′, BCB′ are the principal axes, prove that CA, CB are the bisectors of $V\hat{C}F$. [In Fig. 151, take VP parallel to BC.]

122. If two points on a conic have the same Frégier point (not at infinity), prove that the conic must be a circle.

123. On a chord PQ of a rectangular hyperbola as diameter a circle is described, cutting the curve again at V, V'; prove that the normals at V, V' are parallel to PQ.

124. Given two points on a conic and their two Frégier points, construct the conic.

125. F is the Frégier point of a point V on a given parabola; if V varies, prove that F traces out a coaxal parabola.

126. F_1, F_2 are the Frégier points of two points V_1, V_2 on a conic; prove that the axes of the conic are parallel to the bisectors of the angles between the lines V_1V_2, F_1F_2.

127. F is the Frégier point of a point V on a given central conic; if V varies, prove that F traces out a concentric homothetic conic. Is there any exceptional case?

[If the ordinate and normal of V meet the major axis at N, G, and if C is the centre, use the fact that $\dfrac{NG}{CN}$ is constant.]

128. If a chord of a parabola subtends a right angle at the vertex A and meets the axis at F, prove that AF is equal to the latus rectum.

129. A variable parabola touches a fixed line at a fixed point V and the Frégier point F of V is fixed; prove that the envelope of the parabola is a circle.

130. Prove that circles, whose diameters are parallel chords of a rectangular hyperbola, are coaxal.

131. A normal at a point P on a conic cuts the principal axes at G, g; Q is a point on PG such that $\frac{2}{PQ}=\frac{1}{PG}+\frac{1}{Pg}$; prove that any chord through Q subtends a right angle at P.

132. A parabola, focus S, reciprocated w.r.t. a point C, becomes a conic σ; prove that S becomes the polar of the Frégier point of C w.r.t. σ.

133. PQ, RS are parallel chords of a rectangular hyperbola: DD' is the diameter which bisects them; prove that a common tangent of the circles, on PQ and RS as diameters, subtends a right angle at D and D'.

Desargues' Theorem yields a very neat proof of Pascal's property; an interesting fact, since Pascal ascribes the ideas, which characterise his researches, to the influence of Desargues.

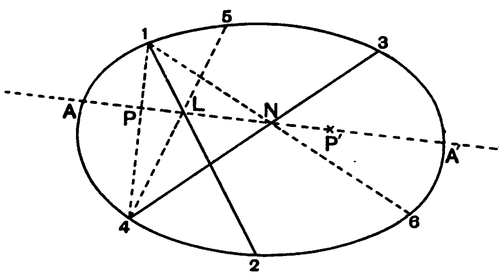

FIG. 152.

Let 123456 be the inscribed hexagon.

Let 12, 45 meet at L and 34, 61 at N.

Join LN and produce it to meet the conic at A, A'.

Let LN meet 14 at P, take a point P' on LN so that the point-pairs L, N; A, A'; P, P' form an involution.

By applying Desargues' theorem to the inscribed quadrangle 1234, it follows that P' lies on 23.

By applying Desargues' theorem to the inscribed quadrangle 1654, it follows that P' lies on 56;

$\therefore$ 23, 56 meet at P';

$\therefore$ the meets of 12, 45; 23, 56; 34, 61 are collinear. Q.E.D.

THEOREM 197.

To construct a conic to pass through three given points and to touch two given lines.

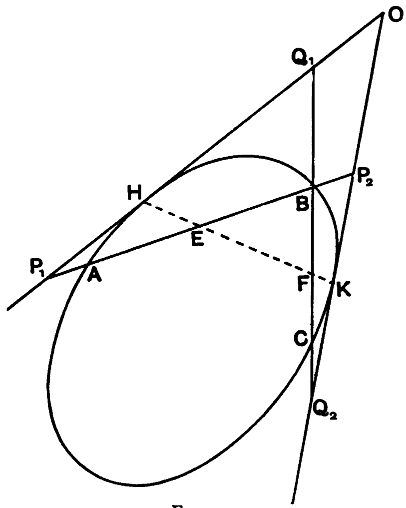

FIG. 153.

Let A, B, C be the given points and OH, OK the given lines.

It is required to determine the points H, K at which the conic touches OH, OK.

Let AB cut OH, OK at P_1, P_2 and HK at E; and let BC cut OH, OK at Q_1, Q_2 and HK at F.

By Theorem 188, E is a double point of the involution determined by A, B; P_1, P_2 and has therefore one of two possible positions which can be constructed, by Theorem 166.

Similarly, F is a double point of the involution determined by B, C; Q_1, Q_2 and has therefore one of two possible positions, which can be constructed.

The line EF which meets OH, OK at the required points H, K, has therefore four possible positions.

Hence we can construct four conics, satisfying the given conditions. Q.E.D.

134. Construct a conic to pass through three given points and have double contact with a given conic.

135. Construct a conic to pass through two given points and touch three given lines.

136. Inscribe in a given triangle a conic having double contact with a given conic.

137. Construct a conic, given four points on it and a pair of points conjugate w.r.t. it.

138. Construct a conic to circumscribe a given triangle and have a given focus.

THEOREM 198.

(1) Every conic through the four common points A, B, C, D of two rectangular hyperbolas is a rectangular hyperbola; and D is the orthocentre of the triangle ABC.

(2) If a rectangular hyperbola circumscribes a triangle ABC, it passes through its orthocentre.

(1) Let ω, ω' be the circular points at infinity.

Then ω, ω' are conjugate points w.r.t. the two given rectangular hyperbolas through A, B, C, D.

$\therefore$ by Theorem 190 (1), ω, ω' are conjugate points w.r.t. every conic through A, B, C, D.

$\therefore$ every such conic is a rectangular hyperbola. Q.E.D.

But AB, CD; AC, BD; AD, BC are three conics of the pencil;

$\therefore$ they are perpendicular line-pairs;

$\therefore$ D is the orthocentre of the triangle ABC. Q.E.D.

(2) Let the perpendicular from A to BC cut the curve in D.

Then the given rectangular hyperbola and the perpendicular line-pair AD, BC determine a pencil of rectangular hyperbolas.

Now AB, CD is a conic of this pencil;

$\therefore$ AB is perpendicular to CD;

$\therefore$ D is the orthocentre of ABC;

$\therefore$ the orthocentre of ABC lies on the rectangular hyperbola.

Q.E.D.

This theorem was published in 1821 by Brianchon and Poncelet in a joint memoir. It is capable of a very simple analytical proof.

139. PQR is a triangle, right-angled at P, inscribed in a rectangular hyperbola; prove that the tangent at P is perpendicular to QR.

140. PQ is a chord of a rectangular hyperbola; the circle on PQ as diameter cuts the curve again at R, S; prove that RS is a diameter of the hyperbola. [Use Ex. 139.]

141. Prove, by reciprocating Theorem 198, Steiner's theorem, viz. the orthocentre of a triangle circumscribing a parabola lies on the directrix.

142. PQR is a self-conjugate triangle w.r.t. a conic S_1; a conic S_2, inscribed in PQR, touches a directrix of S_1; prove that the director circle of S_2 passes through a focus of S_1.

143. Prove that the pedal triangle of a triangle inscribed in a rectangular hyperbola is self-conjugate for the hyperbola.

144. If the normal at a point P on a rectangular hyperbola meets the curve again at Q, prove that the radius of curvature at P equals $\frac{1}{2}$PQ. [Apply Theorem 23, Part I., to the triangle formed by P and two points adjacent to it on the curve.]

145. A variable rectangular hyperbola circumscribes a fixed triangle; find the loci of the poles of the sides of the triangle w.r.t. the hyperbola.

146. If the normal at a point P on a rectangular hyperbola, centre C, meets the curve again at Q, and if R is the mid-point of PQ, prove that $R\hat{C}P = 90°$.

147. A variable equilateral triangle is inscribed in a rectangular hyperbola; find the locus of its centroid.

148. $ABCD$ is a given parallelogram inscribed in a rectangular hyperbola; PH, PK, PL, PM are the perpendiculars from a variable point P on the curve to AB, BC, CD, DA; prove that

$$PH \cdot PL = PK \cdot PM.$$

149. The mid-points of the sides of a variable triangle move on a rectangular hyperbola; find the locus of its circumcentre.

150. Prove that a system of conics, through four fixed points, can be projected into a system of rectangular hyperbolas.

151. HK is a variable chord of a rectangular hyperbola, fixed in direction; PP' is the diameter perpendicular to HK; find the locus of the meet of HP, KP'.

152. Two rectangular hyperbolas intersect at A, B, C, D; prove that the circles whose diameters are AB, CD cut orthogonally.

A system of confocal conics form, by definition, a range of conics; Theorem 189 therefore admits of direct application to a confocal system.

THEOREM 199.

(1) Through any given point T, two conics can be drawn, belonging to a given confocal system; and these conics cut each other orthogonally.

(2) If TX, TY are the tangents to the two confocals through T; if S, H are the foci, and if TP, TQ are the tangents from T to any conic of the system, then TX, TY are the internal and external bisectors of the angles STH and PTQ.

(3) TX, TY are conjugate lines w.r.t. each conic of the system.

(4) The locus of the poles of a given line w.r.t. a system of confocal conics is a line perpendicular to the given line.

(1) Let S, H; S', H' be the foci of the given system; and ω, ω' the circular points at infinity.

Then S, H; S', H'; ω, ω' are the pairs of opposite vertices of a quadrilateral circumscribing each conic of the confocal system.

Let TX, TY be the double lines of the involution, defined by TS, TH; $T\omega$, $T\omega'$.

Then by Theorem 189 Corollary, a conic of the system can be drawn to touch TX at T and similarly a second conic of the system to touch TY at T.

Since TX, TY are the double lines, $T\{XY;\ \omega\omega'\}$ is harmonic;

$$\therefore\ X\hat{T}Y = 90^\circ;$$

$\therefore$ the conics cut orthogonally at T. Q.E.D.

(2) By Theorem 189, TP, TQ are a line-pair of the same involution.

Since the double lines are at right angles, they are the bisectors of the angles between each line-pair of the pencil.

$\therefore$ TX, TY are the bisectors of the angles between TS, TH and TP, TQ. Q.E.D.

(3) Since TX, TY are harmonically conjugate to each pair of tangents from T to the system of conics, they are conjugate lines w.r.t. each conic. Q.E.D.

(4) Let a conic of the system touch the given line TX at T. Let TY be the tangent at T to the second confocal through T.

Then by (1), TX is perpendicular to TY.

But by (3), TY is conjugate to TX w.r.t. each conic of the system;

$\therefore$ the locus of the poles of TX w.r.t. the system is the perpendicular line TY. Q.E.D.

153. Deduce, from Theorem 199, corresponding properties of the parabola.

154. From a point A on a conic S_1, tangents AB, AC are drawn to a confocal conic S_2; prove that AB, AC are equally inclined to the tangent at A to S_1.

155. Given two tangents to a conic and one focus, find the locus of the other focus.

156. Y, Z are the feet of the perpendiculars from a focus S of an ellipse to two tangents TY, TZ; prove that the perpendicular from T to YZ passes through the other focus.

157. S, H are the foci of an ellipse; T is the pole of a chord PQ through S; prove that the normals at P, Q meet on HT.

158. Construct the centre of a conic which has a given focus and is inscribed in a given triangle.

159. $ABCD$ is a given parallelogram; a variable conic touches AB, BC and its real foci lie on AD, DC; prove that the locus of its centre is a straight line.

160. $ABCD$ is a parallelogram circumscribing a conic, focus S; prove that the circles ABS, ADS are equal.

161. $ABCD$ is a fixed parallelogram circumscribing a variable conic σ; prove that the locus of the foci of σ is a rectangular hyperbola through A, B, C, D.

162. One focus of an ellipse inscribed in a triangle is at the orthocentre; prove that the centre of the ellipse is at the nine-point centre.

163. A variable conic is inscribed in a fixed quadrilateral, inscribed in a circle; if one focus lies on the circle, prove that the other focus also lies on the circle.

164. PQ is a diameter of a conic; R is a point on the curve such that PR, QR make equal angles with the tangent at R; prove that the pole of PR lies on the director circle.

165. The sides BC, CA, AB of a triangle ABC, inscribed in a conic S_1, touch a confocal conic S_2 at P, Q, R; prove that the escribed circles of ABC touch the sides at P, Q, R. [Let the tangents at A, B, C to S_1 form the triangle XYZ and prove that X, Y, Z are the excentres: and use Theorem 199 (4).]

THEOREM 200.

The locus of the centres of a pencil of conics, circumscribing a fixed quadrangle, is a conic circumscribing the common self-conjugate triangle of the pencil, and passing through the mid-points of the six sides of the quadrangle, and having its asymptotes parallel to the axes of the two parabolas of the pencil.

The proof is left to the reader.

[In Theorem 192, take l as the line at infinity.]

Another method of proving Theorem 200 is suggested in Ex. 91, Chapter IX.

166. Prove Theorem 200.

167. Prove that the centre of a rectangular hyperbola circumscribing a given triangle lies on the nine-point circle of the triangle.

168. A variable rectangular hyperbola touches a fixed line at a given point and passes through another fixed point; find the locus of its centre.

169. ABC is a triangle inscribed in a rectangular hyperbola, centre O; H is its orthocentre; HO meets the hyperbola at D; prove that A, B, C, D are concyclic. Use Part I, page 43, Ex. 9.]

170. If two conics of a four-point pencil have parallel axes, prove that all the conics have parallel axes, and that one of them is a circle.

171. Prove that the lines joining a point to the vertices of a quadrilateral circumscribing a conic meet the polars of those vertices in points which lie on a conic through the fixed point.

172. A conic passes through two fixed points and has two pairs of fixed points as conjugate points; prove that the locus of its centre is a conic.

173. The conic σ is the locus of the centres of all conics of a pencil through four fixed points; P is any point on σ; prove that the polars of P, w.r.t. conics of the pencil, are parallel.

174. A variable rectangular hyperbola passes through a fixed point A; if its circle of curvature at A is fixed, find the locus of its centre.

175. The conic σ is the locus of the centres of all conics circumscribing the quadrangle ABCD; prove that the centre of σ is the mean centre of the points A, B, C, D.

176. A variable conic circumscribes a trapezium; prove that its centre lies on one of two fixed lines.

177. A variable conic circumscribes a cyclic quadrilateral; prove that the locus of its centre is a rectangular hyperbola.

178. $ABCD$ is a cyclic quadrilateral; prove that the axes of the two parabolas which pass through A, B, C, D intersect at right angles at the centroid of A, B, C, D.

179. Prove that the nine-point circles of the triangles ABC, ACD, ABD, BCD have one common point.

180. Construct the centre of a rectangular hyperbola which passes through two fixed points and touches a given line at a given point.

181. A given line l meets any conic through four fixed points A, B, C, D in P, Q; prove that P, Q are conjugate points w.r.t. the eleven-point conic of the pencil through A, B, C, D corresponding to the line l.

182. A, B, C, D are four points on a hyperbola; prove that the asymptotes of the hyperbola are parallel to a pair of conjugate diameters of the centre-locus of all conics through A, B, C, D.

183. σ is the locus of the centres of all conics through four fixed points; prove that the asymptotes of any one of the conics meet σ at the ends of a diameter.

184. A, B, C, D are four points on a rectangular hyperbola S; prove that the axes of the centre-locus of all conics through A, B, C, D are parallel to the asymptotes of S.

185. Five quadrangles are formed from five points, no three of which are collinear: prove that the five conics which pass through the mid-points of the sides of the quadrangles have one common point.

186. PQ is a diameter of a rectangular hyperbola: any circle through P, Q cuts the hyperbola again at R, S; prove that RS is a diameter of the circle.

187. Two chords AB, CD of a rectangular hyperbola meet at right angles at E; prove that the circle through E and the mid-points of AB and CD passes through the centre of the hyperbola.

188. Prove that the locus of the pole of a fixed line w.r.t. a variable conic, circumscribing a fixed square, is a rectangular hyperbola.

189. A circle meets a rectangular hyperbola at P, Q, R, S; if PP′ is a diameter of the hyperbola, prove that P′ is the orthocentre of QRS.

190. The conic σ is the locus of the poles of a fixed line l w.r.t. a system of conics through four fixed points A, B, C, D; P, Q are the meets of l with σ; prove that σ touches each of the four conics which pass through P, Q and touch AB, BC, CA, and that σ touches twelve other conics formed in a similar way.

191. A rectangular hyperbola circumscribes an equilateral triangle ABC; prove that its centre lies on the incircle of ABC.

192. Prove that the locus of the centre of a conic, passing through the incentre and excentres of a triangle, is the circumcircle of the triangle.

THEOREM 201 [PLÜCKER'S THEOREM].

(1) The three circles, whose diameters are the joins of opposite vertices of a quadrilateral, are coaxal; and the director circles of all conics, inscribed in the quadrilateral, belong to the same coaxal system.

(2) The centres of all conics inscribed in a given quadrilateral lie on a line, passing through the mid-points of the joins of opposite vertices of the quadrilateral.

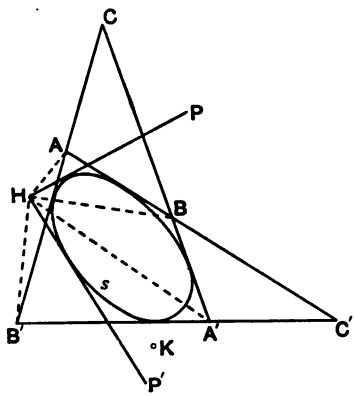

FIG. 154.

(1) Let A, A'; B, B'; C, C' be the pairs of opposite vertices of the quadrilateral. Let the circles on AA', BB' as diameters meet at H, K.

Let PH, HP' be the tangents from H to any conic S inscribed in the quadrilateral.

Then HA, HA'; HB, HB'; HC, HC'; HP, HP' form an involution by Sturm's theorem.

But $A\hat{H}A' = 90° = B\hat{H}B'$, being angles in a semi-circle.

$\therefore$ since two line pairs are at right angles, every line-pair of the involution is at right-angles. [Theorem 173 (2).]

$\therefore$ the circle on CC' as diameter passes through H; and the director circle of S passes through H.

Similarly the circle on CC' as diameter and the director circle of S pass through K.

$\therefore$ the circles on AA', BB', CC' as diameters are coaxal and the director circles of all conics inscribed in the quadrilateral belong to the same coaxal system. Q.E.D.

(2) $\therefore$ the centres of all the director circles lie on a line which passes through the mid-points of AA', BB', CC'.

But the centre of a conic coincides with the centre of its director circle.

$\therefore$ the centres of all conics inscribed in the quadrilateral lie on this line. Q.E.D.

Theorem 201 (1) was discovered independently by Gaskin.

Theorem 201 (2) is due to Newton. A statical proof of it is given in Part I., page 120.

For another method, see Ex. 194.

Theorem 201 is the reciprocal of Theorem 198, as Ex. 193 will show.

193. If H lies on the director circle of each of two conics, prove, by reciprocating w.r.t. H, that it lies on the director circle of every conic touching the four common tangents of the two given conics.

194. Deduce Theorem 201 (2) from Theorem 191 (2).

195. What does Theorem 201 become when the points C, C' coincide in Fig. 154?

196. If a system of conics touch four fixed lines, prove that the radical axis of their director circles is the directrix of the parabola touching the four lines.

197. A conic inscribed in the triangle ABC touches BC at D; if A', D' are the mid-points of BC, AD, prove that the centre of the conic lies on $A'D'$.

198. A, B are the centres of the two rectangular hyperbolas which can be inscribed in a given quadrilateral; prove that any circle through A, B is orthogonal to the director circle of any conic inscribed in the quadrilateral.

199. A variable parabola touches two fixed lines; if its axis is fixed in direction, find the locus of the poles of a fixed line.

200. A variable conic is inscribed in a given triangle; if its director circle pass through one fixed point, prove that it must pass through another fixed point.

201. By taking, in Theorem 201, one side of the quadrilateral as the line at infinity, prove that the orthocentre of a triangle circumscribing a parabola lies on the directrix.

202. Prove that, in general, one and only one conic can be drawn to touch four fixed lines and have its centre on a given line.

203. Prove that the lines bisecting the joins of opposite vertices of each of the five quadrilaterals formed by five straight lines are concurrent.

204. If a parabola is inscribed in the cyclic quadrilateral $ABCD$, prove that AC, BD meet on its directrix.

205. A variable conic has a fixed focus and touches two fixed lines, prove that its director circle passes through two fixed points.

206. If in Ex. 205 A, B are the fixed points and P, Q the reflections of the fixed focus in the fixed lines, prove that P, Q lie on AB.

207. Prove that the circumcircle of a triangle self-conjugate w.r.t. a conic is orthogonal to the director circle of the conic.

208. Prove that the centre of a conic is the radical centre of the circumcircles of all triangles self-conjugate w.r.t. the conic.

209. $ABCD$ is a quadrilateral circumscribing a parabola; prove that the join of the mid-points of AC, BD is parallel to the axis of the parabola.

210. Four conics circumscribe the triangle ABC and have a common focus D; prove that the director circle of any conic, touching the four corresponding directrices, passes through D.

211. Prove that any two tangents to a central conic and the four perpendiculars to them from the foci touch a conic.

212. Show how to draw a rectangular hyperbola to touch four given lines.

213. $ABCD$ is a quadrangle inscribed in a conic; P, P'; Q, Q'; R, R' are the opposite vertices of the quadrilateral formed by the tangents at A, B, C, D. If the conic σ is the locus of the centres of all conics through A, B, C, D, prove that the join of the mid-points of PP', QQ' is a tangent to σ.

214. A parabola is drawn to touch the four common tangents of two rectangular hyperbolas; prove that its directrix is the perpendicular bisector of the join of the centres of the hyperbolas.

215. Given a triangle ABC and the centre of an ellipse inscribed in it, construct the points of contact with the sides.

216. T is the pole of the chord PQ of a parabola; prove that the directrix of the parabola is halfway between T and the polar of T w.r.t. the circle on PQ as diameter.

217. Prove that the director circles of a system of conics touching four fixed lines cut any transversal in an involution range.

218. A variable conic touches the fixed lines AB, AC at fixed points B, C; prove that its director circle belongs to a fixed coaxal system, of which A is a limiting point.

219. Prove that the circumcircle of the diagonal line triangle of a quadrilateral cuts the line, passing through the mid-points of the joins of opposite vertices, at the centres of the rectangular hyperbolas which are inscribed in the quadrilateral.

THEOREM 202.

E, F, G, H are the feet of the four normals from a point *O* to a conic, centre *C*; *M, N* are the feet of the perpendiculars from the pole *T* of *EF* to the axes *ACA′, BCB′*; if *GH* cuts *CA CB* at *M′, N′*, then $M'C = CM$ and $N'C = CN$.

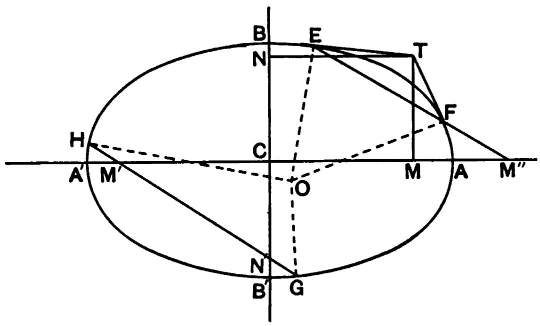

FIG. 155.

Let *EF* meet *CA* at *M″*.

By Theorem 146, the points *E, F, G, H* lie on a rectangular hyperbola passing through *C* and having its asymptotes parallel to *CA, CB*.

Now the transversal *AA′* is cut in involution by all conics through *EFGH*; and point-pairs of this involution are *M′, M″*; *A, A′*; *C*, ∞.

$\therefore$ *C* is the centre of this involution.

$$\therefore CM' . CM'' = CA . CA' = -CA^2.$$

But, since *T* is the pole of *EF*, $\therefore$ *TM* is the polar of *M″*, and $\{AA'; MM''\}$ is harmonic.

$$\therefore CM . CM'' = CA^2;$$

$$\therefore CM = -CM';$$

$\therefore M'C = CM$ and similarly $N'C = CN$. Q.E.D.

THEOREM 203 [JOACHIMSTHAL'S THEOREM].

The circle drawn through the feet of three of the normals from a point to a conic cuts the conic again at the point, which is diametrically opposite to the foot of the fourth normal.

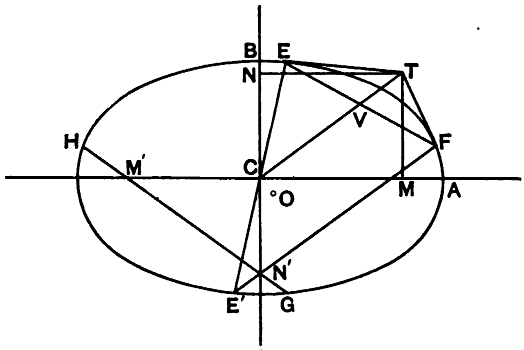

FIG. 156.

The notation is the same as in Theorem 202.

Let EC meet the conic again at E' and CT meet EF at V.

Since $EV = VF$ and $EC = CE'$, CVT is parallel to $E'F$.

Also by Theorem 202, MN is parallel to $M'N'$ or GH.

But CT, MN are equally inclined to CA, being the diagonals of a rectangle.

$\therefore$ $E'F$, GH are equally inclined to CA, an axis of the conic.

$\therefore$ the circle through FGH cuts the conic again at E'. [See Ch. VI. Ex. 125.] Q.E.D.

220. [HARVEY'S THEOREM.] **If P, Q, R are the feet of the normals to a parabola from any point, prove that the circumcircle of the triangle PQR passes through the vertex of the parabola.**

221. **P is a fixed point on a central conic; Q is a variable point on the normal at P; L, M, N are the feet of the other normals from Q to the conic; prove that the sides of the triangle LMN envelope a parabola.**

222. P, Q, R, S are the feet of the normals from a variable point to a conic; if PQ passes through a fixed point, prove that RS touches a fixed conic.

223. PQ is a chord of a parabola fixed in direction; prove that the normals at P and Q meet on a fixed line.

224. The normals at the extremities of a variable chord PQ of a conic, centre C, meet on a fixed normal l of the conic; prove that PQ envelopes a parabola, whose directrix passes through C and whose focus is the foot of the perpendicular from C to the tangent, perpendicular to l.

225. The normals at the points P, Q, R, S on a conic are concurrent, and the circles QRS, RSP, SPQ, PQR meet the conic again at p, q, r, s respectively; prove that the normals at p, q, r, s are concurrent.

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226. EFG is the diagonal point triangle of the quadrangle $ABCD$; P is any point; EP_1, FP_2, GP_3 are the harmonic conjugates of EP, FP, GP w.r.t. the sides of the quadrangle which meet at E, F, G respectively; prove that EP_1, FP_2, GP_3 are concurrent.

227. A variable conic, centre P, passes through four fixed points; Q is a point on the conic, the tangent at which has a fixed direction; prove that PQ passes through a fixed point.

228. ABC is a triangle inscribed in a conic; PQ, PR are two chords parallel to CB, CA; PQ meets AB, AC at L, U; PR meets BC, BA at V, M; prove that $\frac{LU}{UQ}=\frac{RV}{VM}$. [Prove that $\{LUQ\infty\}=\{RVM\infty\}$.]

229. Two conics have double contact at A, B; prove that the polars of any point P, w.r.t. the conics, meet on AB.

230. Obtain, from Theorem 288, a construction for Theorem 165, using a ruler only.

CHAPTER XII.

MISCELLANEOUS PROPERTIES.

THEOREM 204.

(1) If two triangles circumscribe a conic, their six vertices lie on another conic.

(2) If two triangles are inscribed in a conic, their six sides touch a conic.

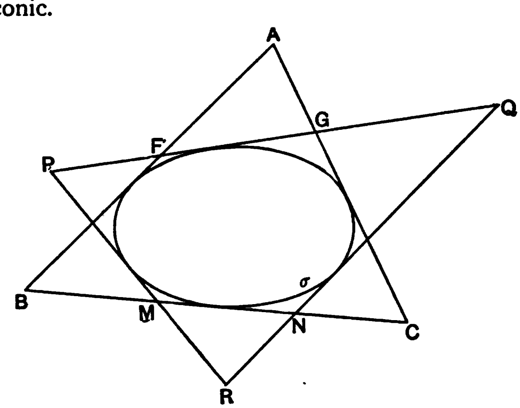

FIG. 157.

(1) Let ABC, PQR be two triangles circumscribing the conic σ. Let AB, AC cut PQ in F, G and RP, RQ cut BC in M, N.

Then $\qquad A\{PBCQ\} = A\{PFGQ\} = \{PFGQ\}$

$= \{MBCN\}$, by Theorem 57,

$= R\{MBCN\}$

$= R\{PBCQ\}$;

$\therefore$ by Theorem 74, A, R, P, B, C, Q lie on a conic. Q.E.D.

(2) The proof is left to the reader.

[It may be proved by reciprocating (1) or by writing at full length the dual of the method of (1) or by a "reductio ad absurdum" method.]

THEOREM 205.

The circumcircle of a triangle, formed by three tangents to a parabola, passes through the focus of the parabola.

Let ABC be the triangle formed by the three tangents: let S be the focus of the parabola, and ω, ω' the circular points at infinity.

Then the triangles ABC, $S\omega\omega'$ circumscribe the parabola.

$\therefore$ by Theorem 204 (1), A, B, C, S, ω, ω' lie on a conic;

$\therefore$ A, B, C, S lie on a circle. Q.E.D.

THEOREM 206.

If two conics are such that one triangle can be inscribed in one and circumscribed to the other, then an unlimited number of such triangles exist.

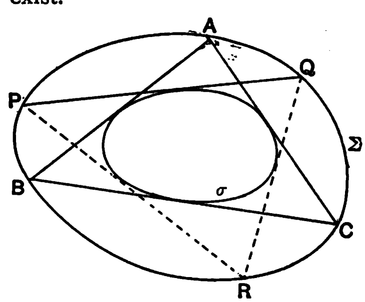

FIG. 158.

Let ABC be the given triangle which circumscribes the conic σ and is inscribed in the conic Σ.

Let any tangent to σ cut Σ at P, Q and let the other tangents from P, Q to σ meet at R.

By Theorem 204 (1), A, B, C, P, Q, R lie on a conic.

Now only one conic can be drawn through five given points [Th. 147].

$\therefore$ since Σ passes through A, B, C, P, Q, it must also pass through R;

$\therefore$ PQR is inscribed in Σ and circumscribed to σ;

$\therefore$ an unlimited number of such triangles exist. Q.E.D.

1. Prove Theorem 204 (2) by the dual method.

2. Prove Theorem 204 (2) by the "reductio ad absurdum" method.

3. The in-centre of a triangle inscribed in a rectangular hyperbola lies on the curve; prove that the in-circle passes through the centre of the hyperbola.

4. T_1, T_2 are the poles of two chords P_1Q_1, P_2Q_2 of a conic; prove that the six points T_1, P_1, Q_1, T_2, P_2, Q_2 lie on a conic.

5. A variable parabola has a fixed self-conjugate triangle; prove that the locus of its focus is the nine-point circle of the triangle.

6. Through each of two points P, Q, a pair of lines are drawn so as to form a cyclic quadrilateral $ABCD$; prove that the focus of the parabola inscribed in $ABCD$ lies on PQ.

7. If two circles s_1, s_2; radii r_1, r_2; centres A_1, A_2; are such that triangles can be inscribed in s_1 and circumscribed to s_2, prove that $-2r_1r_2$ is equal to the power of A_2 w.r.t. s_1.

8. PQR is a triangle circumscribing a conic, focus S; if P, Q, R, S are concyclic, prove that the conic is a parabola.

9. F is a focus of an ellipse S_1; S_2 is a circle, centre F, of radius equal to the major axis; prove that an unlimited number of triangles can be inscribed in S_2 and circumscribed to S_1.

10. With the notation of Ex. 9, prove that the orthocentre of every triangle inscribed in S_2 and circumscribed to S_1 is at the second focus of S_1.

11. If two conics are such that one quadrilateral can be inscribed in one and circumscribed to the other, prove that an unlimited number of such quadrilaterals exist. [If quadrilateral $ABCD$ is inscribed in Σ_1 and circumscribed to Σ_2, project Σ_1 into a circle σ_1 having projection of meet of AC, BD as centre: and note that σ_1 is director circle of projection of Σ_2.]

12. If $ABCD$ is a variable quadrilateral inscribed in one fixed conic and circumscribed about another fixed conic, prove that AC, BD meet at a fixed point.

13. A variable parabola has a fixed focus and touches a fixed line l; prove that the locus of the point of contact of the other tangent from a fixed point on l is a circle.

14. T is the pole of a chord PQ of a parabola, focus S; prove that TQ touches the circle through T, P, S.

15. Construct the focus and directrix of the parabola which touches four given lines.

16. T is the pole of a chord PQ of a parabola; R is the mid-point of PQ; a hyperbola is drawn through P, Q, T with one asymptote

parallel to TR; if it meets the parabola again at S, prove that the tangent at S to the parabola is parallel to the other asymptote.

17. T is the pole of a chord PQ of a hyperbola S_1, centre C; a hyperbola S_2 is drawn through T, P, Q with asymptotes parallel to those of S_1; prove that S_2 passes through C and that its centre lies on CT.

18. Find the locus of the focus of a parabola touching a given line at a given point and a second fixed line.

19. A conic is inscribed in a triangle ABC and passes through the circumcentre of ABC; prove that the circumcircle touches the director circle of the conic.

20. T is the pole of a chord PQ of a parabola, focus S; O is the circumcentre of TPQ; prove that $O\hat{S}T=90°$.

THEOREM 207.

If two triangles are self-conjugate w.r.t. a conic, then their six sides touch a conic: and their six vertices lie on a conic.

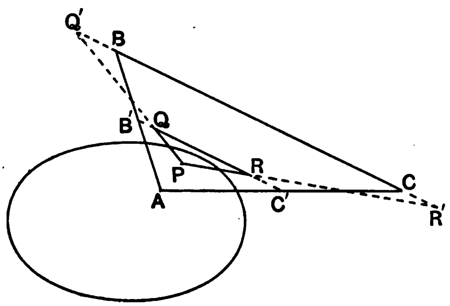

FIG. 159.

Let ABC, PQR be two triangles, self-conjugate w.r.t. a conic.

Let PQ, PR meet BC in Q', R' and AB, AC meet QR in B', C'.

Now Q' is the meet of PQ, BC, whose poles are R, A;

$\therefore$ the polar of Q' is AR; and similarly the polar of R' is AQ.

$$\begin{aligned}\therefore \{BQ'R'C\} &= A\{CRQB\}, \text{ the pencil of polars, by Th. 55,}\\ &= \{C'RQB'\}\\ &= \{B'QRC'\};\end{aligned}$$

$\therefore$ BB', QQ', RR', CC' touch a conic, touching BC, $B'C'$, by Th. 75;

$\therefore$ the sides of ABC, PQR touch a conic; Q.E.D.

$\therefore$ by Theorem 204, the vertices of ABC, PQR lie on a conic. Q.E.D.

THEOREM 208.

If two triangles are inscribed in the same conic, there exists a conic w.r.t. which both triangles are self-conjugate.

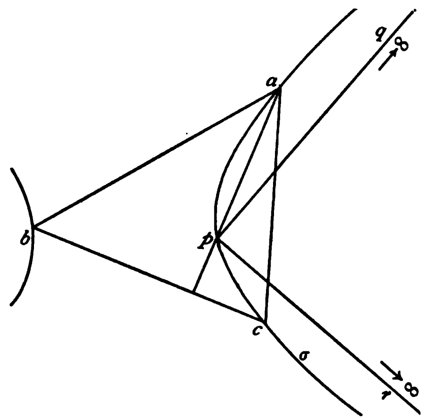

FIG. 160.

Let ABC, PQR be the two triangles inscribed in a conic Σ.

Project QR to infinity and P into the orthocentre of the projection of the triangle ABC. [Theorem 48.]

Denote corresponding elements of the new figure by small letters.

The conic σ circumscribes the triangle abc and passes through its orthocentre p; $\therefore$ σ is a rectangular hyperbola. [Th. 198.]

But q, r are points at infinity on σ;

$$\therefore\ q\hat{p}r = 90^\circ;$$

$\therefore$ pqr is a self-conjugate triangle w.r.t. any circle, centre p.

Let σ_1 be the circle w.r.t. which the triangle abc is self-conjugate.

Then the centre of σ_1 is at p: and therefore pqr is self-conjugate w.r.t. σ_1.

Hence abc and pqr are self-conjugate w.r.t. σ_1;

$\therefore$ in the original figure, there exists a conic Σ_1 w.r.t. which ABC and PQR are each self-conjugate. Q.E.D.

Corollary.

If two triangles circumscribe the same conic, there exists a conic w.r.t. which both triangles are self-conjugate.

This follows at once from Theorem 204.

For another method of proof of Theorem 208, see Ex. 25, 26.

THEOREM 209.

If two conics S_1, S_2 are such that one triangle can be inscribed in S_1, which is self-conjugate to S_2, then an unlimited number of such triangles exist.

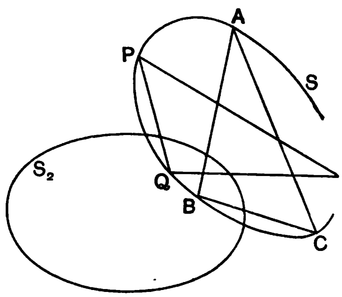

FIG. 161.

Let ABC be the given triangle inscribed in S_1, which is self-conjugate to S_2.

Take any point P on S_1: let the polar of P w.r.t. S_2 cut S_1 at Q; let the line through P conjugate to PQ w.r.t. S_2 cut the polar of P at R.

Then PQR is a self-conjugate triangle w.r.t. S_2;

$\therefore$ by Theorem 207, A, B, C, P, Q, R lie on a conic.

But five points determine a conic, and A, B, C, P, Q lie on S_1.

$\therefore$ R lies on S_1;

$\therefore$ PQR is inscribed in S_1 and self-conjugate to S_2;

$\therefore$ an unlimited number of such triangles exist. Q.E.D.

THEOREM 210.

If two conics S_1, S_2 are such that one triangle can be circumscribed about S_1, which is self-conjugate w.r.t. S_2, then an unlimited number of such triangles exist.

The proof is left to the reader.

Definitions.

(1) The conic S_1 is said to be **harmonically circumscribed** to the conic S_2, if S_1 circumscribes one triangle (and therefore an unlimited number of triangles), self-conjugate w.r.t. S_2.

(2) The conic S_1 is said to be **harmonically inscribed** to the conic S_2, if S_1 is inscribed in one triangle (and therefore an unlimited number of triangles), self-conjugate w.r.t. S_2.

THEOREM 211.

If a conic S_1 is harmonically circumscribed to a conic S_2, then S_2 is harmonically inscribed to S_1.

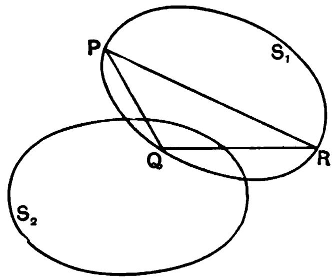

FIG. 162.

PQR is a triangle inscribed in S_1 and self-conjugate w.r.t. S_2.

Let S be a conic w.r.t. which S_1 and S_2 are reciprocal. [Theorem 111.]

Reciprocate the figure w.r.t. S, and consider, in one system, the new figure and the old figure.

Then, S_1 becoming S_2, and S_2 becoming S_1, PQR becomes a triangle pqr circumscribing S_2 and self-conjugate to S_1;

$\therefore$ S_2 is harmonically inscribed to S_1. Q.E.D.

THEOREM 212.

If PQR is a triangle circumscribing a rectangular hyperbola S, the polar circle σ of PQR passes through the centre C of S.

Let ω, ω' denote the circular points at infinity.

Now by hypothesis, S is harmonically inscribed in σ;

$\therefore$ σ is harmonically circumscribed about S.

But $C\omega\omega'$ is a self-conjugate triangle w.r.t. S and two of its vertices ω, ω' lie on σ; $\therefore$ C lies on σ. Q.E.D.

This is a special case of a more general theorem. [Th. 215.]

THEOREM 213.

If triangles can be inscribed in a conic S_1 and circumscribed to a conic S_2, prove that all such triangles are self-conjugate to a conic S w.r.t. which S_1 and S_2 are reciprocal.

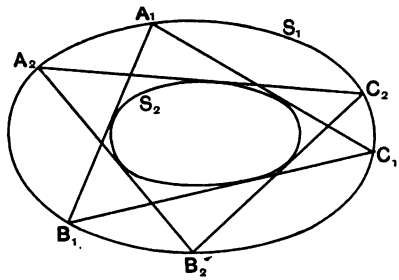

FIG. 163.

Let $A_1B_1C_1$, $A_2B_2C_2$ be two of the triangles.

Let S be the conic w.r.t. which both triangles are self-conjugate. [Theorem 208.]

Reciprocate the system w.r.t. S.

Then $A_1B_1C_1$, $A_2B_2C_2$, being self-conjugate w.r.t. S, reciprocate into themselves;

$\therefore$ S_1 reciprocates into a conic touching the sides of the triangles $A_1B_1C_1$, $A_2B_2C_2$, *i.e.* into S_2;

$\therefore$ S is a conic w.r.t. which S_1 and S_2 are reciprocal.

Let BC be the polar w.r.t. S of *any* point A on S_1;

$\therefore$ BC touches S_2.

Let B be one meet of BC and S_1.

Then the polar of B w.r.t. S touches S_2 and passes through A; let it cut BC at C.

Now the triangles $A_1\ B_1\ C_1$, ABC are self-conjugate w.r.t. S;
$\therefore$ their six vertices lie on a conic.

But A_1, B_1, C_1, A, B lie on S_1;

$\therefore$ C lies on S_1.

Similarly AB touches S_2;

$\therefore$ ABC is a triangle, inscribed in S_1 and circumscribed to S_2 and self-conjugate to S;

But A is any point on S_1;

$\therefore$ all triangles inscribed in S_1 and circumscribed to S_2 are self-conjugate w.r.t. S. Q.E.D.

Corollary.

If a triangle ABC is inscribed in a conic S_1 and circumscribed to a conic S_2, and if PQR is the common self-conjugate triangle of S_1 and S_2, then the vertices of the triangles ABC, PQR lie on a conic.

[PQR is self-conjugate w.r.t. S, Theorem 111 Cor.]

21. Prove Theorem 210.

22. Prove Theorem 213 Corollary.

23. If the polar circle of a triangle circumscribing a conic passes through the centre of the conic, prove that the conic must be a rectangular hyperbola.

24. PQR is a triangle self-conjugate w.r.t. a rectangular hyperbola, centre C; prove that P, Q, R, C lie on a circle.

25. ABC is a given triangle, P is a given point, QR is a given line; prove that there exists a conic Σ w.r.t. which ABC is self-conjugate, and such that P is the pole of QR w.r.t. Σ; and that Σ may be constructed as follows: BC meets QR, AP at Q, Q'; E, F are the double points of the involution B, C; Q, Q'; PE meets QR at R; D is the harmonic conjugate of E w.r.t. P, R. Then Σ is the conic passing through D and touching AE, AF at E, F.

26. Deduce Theorem 208 from Ex. 25 and Theorem 207.

27. ABC, DEF are two self-conjugate triangles w.r.t. a conic S; prove that the polar w.r.t. S of the meet of AD, BE is a Pascal line of the hexagon whose vertices, in some order, are A, B, C, D, E, F; hence prove that the three Pascal lines through the meets of (1) BC, EF; AC, DF and (2) AC, DE; AB, EF and (3) AB, DF; BC, DE are concurrent.

28. The sides of a triangle touch a conic S_1 at A, B, C and its vertices lie on a conic S_2; if PQR is the common self-conjugate triangle of S_1, S_2, prove that A, B, C, P, Q, R lie on a conic. [Use Th. 213.]

29. PQR is a triangle self-conjugate w.r.t. a conic, centre C; prove that the asymptotes of any hyperbola through P, Q, R, C are parallel to a pair of conjugate diameters of the given conic.

30. T is the pole of a chord PQ of a conic S_1; S_2 is a conic through T touching PQ at P; prove that S_1 is harmonically inscribed in S_2.

31. CP, CD are conjugate semi-diameters of a conic S_1; S_2 is a hyperbola through C with its asymptotes parallel to CP, CD; prove that S_1 is harmonically inscribed in S_2.

32. AB is the diameter of a circle S_1; S_2 is a conic, having AB as directrix and its corresponding focus on S_1; prove that S_2 is harmonically inscribed to S_1.

33. The centre of a circle S_1 lies on a rectangular hyperbola S_2; prove that S_1 is harmonically inscribed to S_2.

34. Prove that any circle, whose centre lies on the directrix of a parabola, is harmonically circumscribed to the parabola.

35. O is a point on a rectangular hyperbola S_1, centre C; S_2 is the circle, centre O, radius OC; S_3 is the parabola, whose focus is C and directrix is the tangent at O to S_1; prove that S_1, S_2, S_3 are so related that triangles can be inscribed in any one, which are self-conjugate to any other and are circumscribed to the third.

36. Any circle through the focus of a parabola circumscribes triangles in which the parabola is inscribed.

37. Any circle through the centre of a rectangular hyperbola is harmonically circumscribed to the hyperbola.

38. The orthocentre H of a triangle ABC inscribed in a conic S lies on the director circle of S; prove that the polar of H w.r.t. S touches the circle w.r.t. which ABC is self-conjugate.

39. The conic S_1 is inscribed in a triangle self-conjugate to the conic S_2; if the join of two of its points of contact touches S_2, prove that the joins of the other points of contact also touch S_2.

40. PQ is a chord of a rectangular hyperbola, centre C; prove that the pole of PQ lies on the circle through C, which touches PQ at P.

41. T is the pole of a chord PQ of a conic S; ABC is a self-conjugate triangle w.r.t. S; prove that any conic through A, B, C, T cuts PQ harmonically.

42. The focus S of a conic σ_1 lies on the director circle of a conic σ_2; if σ_1 is harmonically inscribed to σ_2, prove that σ_1 touches the polar of S w.r.t. σ_2.

THEOREM 214 [GASKIN'S THEOREM].

The circumcircle of a triangle, self-conjugate w.r.t. a conic, is orthogonal to the director circle of the conic.

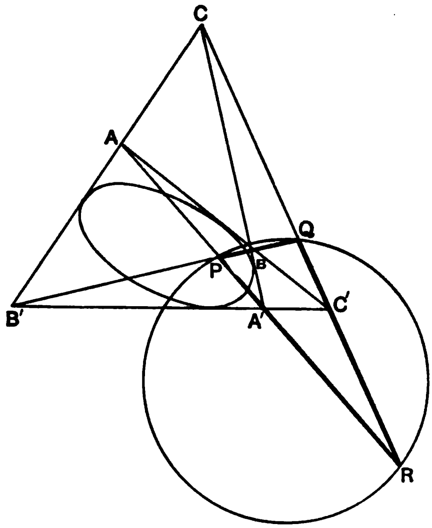

FIG. 164.

Let PQR be the self-conjugate triangle.

Circumscribe a quadrilateral about the conic, having PQR as diagonal line triangle. This can be done by Theorem 73.

Let A, A'; B, B'; C, C' be the pairs of opposite vertices of the quadrilateral.

Since $\{AA'; PR\}$, $\{BB'; PQ\}$, $\{CC'; QR\}$ are harmonic, the circles, whose diameters are AA', BB', CC', are orthogonal to the circle PQR.

$\therefore$ the circle PQR is orthogonal to every circle coaxal with these circles, and therefore, by Theorem 201, is orthogonal to the director circle of the given conic Q.E.D.

This theorem was discovered independently, eight years later, by a Frenchman, M. Faure, in 1860. For another method of proof, see page 349.

43. Prove that the circumcentre of a triangle, self-conjugate to a parabola, lies on the directrix.

44. Prove that the circumcircle of a triangle, self-conjugate to a rectangular hyperbola, passes through the centre of the hyperbola.

45. If two circles are harmonically circumscribed to a conic, prove that their radical axis passes through the centre of the conic.

46. H is the orthocentre of a variable triangle ABC, circumscribing a given conic; AH meets BC at D; if $HA.HD$ is constant, prove that the locus of H is a circle.

47. S is a focus of a conic σ inscribed in a triangle PQR; a rectangular hyperbola is drawn through S, having PQR as a self-conjugate triangle; prove that it touches the major axis of σ.

48. Two conics have double contact at A, B, and are each harmonically circumscribed to a conic S; prove that AB touches S.

49. The circumcircle of a triangle, self-conjugate w.r.t. a given conic, is of fixed size; find the locus of its centre.

50. Two triangles are inscribed in concentric circles and their six vertices lie on a conic; prove that there exists a parabola w.r.t. which both triangles are self-conjugate.

51. Two conics S_1, S_2 have a common focus S, and each is harmonically inscribed to a conic S_3; prove that the meet of two common tangents of S_1, S_2 lies on the polar of S w.r.t. S_3.

52. P is any point on the directrix of a conic, focus S; PT is the tangent from P to the director circle; prove that $PT=PS$.

53. A rectangular hyperbola is harmonically circumscribed to a parabola; prove that the axis of the parabola is parallel to the polar of the focus of the parabola w.r.t. the hyperbola.

54. PQR is a variable triangle self-conjugate w.r.t. a given parabola; if P is fixed, prove that the circle PQR belongs to a fixed coaxal system.

THEOREM 215.

The polar circle S of a triangle circumscribing a conic σ is orthogonal to the director circle of σ.

By hypothesis, σ is harmonically inscribed in S.

$\therefore$ by Theorem 211, S is harmonically circumscribed to σ;

i.e. S circumscribes a triangle, self-conjugate to σ.

$\therefore$ by Theorem 214, S is orthogonal to the director circle of σ.

Q.E.D.

THEOREM 216 [STEINER'S THEOREM].

The orthocentre of a triangle circumscribing a parabola lies on the directrix.

The polar circle of the triangle is, by Theorem 215, orthogonal

to the directrix of the parabola, since the director circle of a parabola is the directrix and the line at infinity.

$\therefore$ the centre of the polar circle of the triangle lies on the directrix;

i.e. the orthocentre of the triangle lies on the directrix.

Q.E.D.

55. Deduce Steiner's theorem from Theorem 201, by taking one side of the quadrilateral as the line at infinity.

56. Deduce Theorem 198 from Theorem 216, by reciprocation.

57. a, b are the axes of a conic inscribed in a given triangle; find the centre of the conic for which a^2+b^2 is a minimum.

58. If a triangle is self-conjugate w.r.t. a rectangular hyperbola, prove that its incentre and excentres lie on the curve.

59. PT, QT are tangents to a parabola; P is the point of contact of PT; PQ is drawn perpendicular to TQ and meets the directrix at D; prove that $D\hat{T}P=90°$.

60. Find the focus and directrix of a parabola which touches two given lines at given points.

61. The vertices of two triangles, having a common orthocentre, lie on a conic; prove that they have the same polar circle.

62. A conic is inscribed in a triangle, self-conjugate w.r.t. a rectangular hyperbola, and has one focus at the centre of the hyperbola; prove that it must be a parabola.

63. G is the centroid of a triangle ABC circumscribing a parabola; prove that the polar of G w.r.t. the parabola touches the conic which passes through A, B, C, and has its centre at G.

64. Given a self-conjugate triangle of a rectangular hyperbola, prove that the locus of its centre is a circle.

65. The polar circle of a triangle circumscribing a conic passes through a focus; prove that its orthocentre lies on a directrix of the conic.

66. A variable conic σ circumscribes a fixed triangle and is harmonically circumscribed to a fixed conic Σ; if the fixed triangle is not self-conjugate w.r.t. Σ, prove that σ passes through another fixed point.

67. A variable triangle circumscribes a fixed conic; if its orthocentre is fixed, prove that its polar circle is also fixed.

68. Prove that the locus of the centres of rectangular hyperbolas, inscribed in a fixed triangle, is a circle.

69. A variable conic is inscribed in a fixed triangle; if the sum of the squares of its axes is constant, find the locus of its centre.

70. Two circles S_1, S_2 are harmonically circumscribed to a conic σ; prove that any circle coaxal with S_1, S_2 is harmonically circumscribed to σ. Generalise this theorem.

71. Two circles S_1, S_2 are harmonically circumscribed to a conic σ; prove that the limiting points of the coaxal system, defined by S_1, S_2, lie on the director circle of σ.

72. BE, CF are altitudes of the triangle ABC; EF meets BC at H; prove that the focus of the parabola inscribed in ABC, and touching EF, lies on AH.

73. Find the locus of the circumcentre of the triangle formed by two fixed tangents, and one variable tangent of a parabola.

74. σ_1 is a parabola harmonically inscribed to a hyperbola σ_2; prove that the asymptotes of σ_2 are conjugate lines w.r.t. σ_1.

75. A system of conics touch three fixed lines; prove that their director circles have a common radical centre.

THEOREM 217.

Pairs of points P, P'; Q, Q'; R, R' divide harmonically the joins of opposite vertices AA'; BB'; CC' of a quadrilateral; then the six points P, P', Q, Q', R, R' lie on a conic.

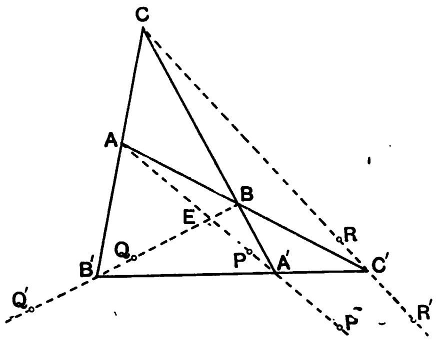

FIG. 165.

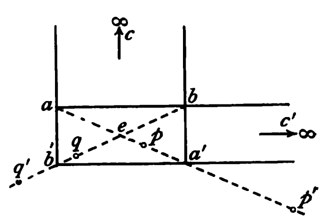

FIG. 166.

Project R, R' into the circular points at infinity.

Then $ABA'B'$ becomes a rectangle, since $B\{AA'; RR'\}$ is harmonic.

In the projected figure,

since $\{aa'; pp'\}$ and $\{bb'; qq'\}$ are harmonic,

$$ep \,.\, ep' = ea^2 = eb^2 = eq \,.\, eq';$$

$\therefore$ q, q', p, p' lie on a circle;

$\therefore$ in the original figure, P, P', Q, Q', R, R' lie on a conic.

Q.E.D.

THEOREM 218 [HESSE'S THEOREM].

If two pairs of opposite vertices of a quadrilateral are conjugate points w.r.t. a conic Σ, then the third pair of opposite vertices are also conjugate w.r.t. Σ.

In Fig. 165, let A, A' and B, B' be conjugate w.r.t. Σ.

Let Σ cut AA'; BB'; CC' at P, P'; Q, Q'; R, R'.

By hypothesis $\{AA'; PP'\}$ and $\{BB'; QQ'\}$ are harmonic.

But only one conic can be drawn through P, P', Q, Q', R.

$\therefore$ by Theorem 217, $\{CC'; RR'\}$ is harmonic.

$\therefore$ C, C' are conjugate w.r.t. Σ. Q.E.D.

THEOREM 219.

If two pairs of opposite sides of a quadrangle are conjugate lines w.r.t. a conic Σ, then the third pair of opposite sides are also conjugate w.r.t. Σ.

This is simply the reciprocal of Hesse's Theorem.

Another method of proof is given on page 348.

THEOREM 220.

If two triangles are conjugate to each other w.r.t. a conic, the meets of corresponding sides are collinear and the joins of corresponding vertices are concurrent.

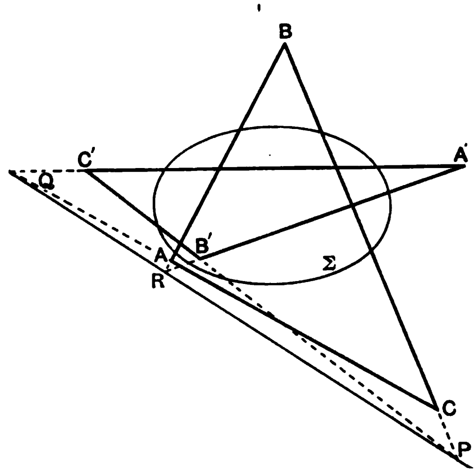

FIG. 167.

Let the triangles ABC, $A'B'C'$ be conjugate to each other w.r.t. the conic Σ (*i.e.* the sides of either are the polars of the vertices of the other w.r.t. Σ).

Let P, Q, R be the meets of BC, $B'C'$; CA, $C'A'$; AB, $A'B'$.

Now in the quadrilateral $ACPR$, two pairs of opposite vertices A, P; C, R are conjugate points w.r.t. Σ;

$\therefore$ the third pair of opposite vertices are conjugate w.r.t. Σ;

$\therefore$ B is conjugate to the meet of AC, PR.

But the polar of B is $A'C'$;

$\therefore$ $A'C'$ passes through the meet of AC, PR;

$\therefore$ Q lies on PR;

$\therefore$ the meets of corresponding sides are collinear;

$\therefore$ also, by Theorem 47, the joins of corresponding vertices are concurrent. Q.E.D.

Another method of proof is suggested in Ex. 76.

76. Prove Theorem 220, by projecting Σ into a circle, having the projection of A as centre.

77. In Theorem 220, prove that the meet of AA′, BB′, CC′ is the pole of PQR w.r.t. Σ.

78. A, A'; B, B'; C, C' are points on a conic; the double points of the involution ranges on the conic, determined by BB', CC'; CC', AA'; AA', BB', are P, P'; Q, Q'; R, R' respectively. Prove that the involutions, determined by AA', PP'; BB', QQ'; CC', RR', have a common pair of corresponding points.

THEOREM 221.

If two conics S_1, S_2 are harmonically circumscribed to a conic Σ, then every conic of the pencil, determined by S_1, S_2, is also harmonically circumscribed to Σ.

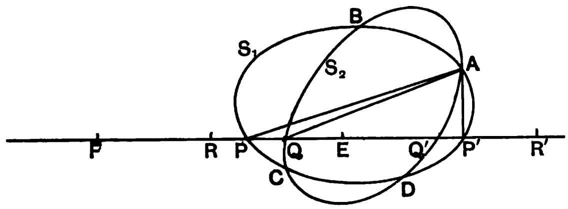

FIG. 168.

Let S_1, S_2 meet at A, B, C, D.

Let the polar of A w.r.t. Σ cut S_1 at P, P', and S_2 at Q, Q', and Σ at E, F, and any other conic S_3 of the pencil at R, R'.

By hypothesis, APP', AQQ' are self-conjugate triangles w.r.t. Σ;

$\therefore$ $\{PP'; EF\}$, $\{QQ'; EF\}$ are harmonic;

$\therefore$ E, F are the double points of the involution, determined by P, P'; Q, Q'.

But by Desargues' theorem, R, R' is a point-pair of this involution.

$\therefore$ $\{RR'; EF\}$ is harmonic;

$\therefore$ ARR' is a self-conjugate triangle w.r.t. Σ;

$\therefore$ S_3 is harmonically circumscribed to Σ. Q.E.D.

By taking S_1, S_2 as line-pairs, we have the following important particular case of Theorem 221:

If two pairs of opposite sides of a quadrangle are conjugate lines w.r.t. a conic, so also is the third pair.

This is the dual of Hesse's theorem: see Theorems 218, 219.

THEOREM 222.

If two conics S_1, S_2 are harmonically inscribed to a conic Σ, then every conic of the range, determined by S_1, S_2, is also harmonically inscribed to Σ.

The proof is left to the reader.

79. Prove Theorem 222.

80. Deduce Hesse's theorem from Theorem 222.

81. ABC, PQR are two triangles inscribed in a conic; two conics are drawn circumscribing ABC and PQR respectively and having double

contact; prove that their chord of contact touches the conic w.r.t. which ABC and PQR are self-conjugate.

82. S_1, S_2 are two conics with parallel asymptotes, and are each harmonically circumscribed to a conic S_3; prove that the common chord of S_1, S_2 passes through the centre of S_3.

83. What is the special case of Ex. 82, if S_1 is a circle?

84. If two confocal conics are harmonically inscribed in a third conic S, prove that S must be a rectangular hyperbola.

85. Two parabolas are inscribed in a triangle ABC and are harmonically inscribed to a conic S; if D is the centre of S, prove that AD, BC are conjugate w.r.t. S.

86. Two conics, which are harmonically circumscribed to a conic S, have double contact at A, B; prove that AB touches S.

87. TP, TQ are the common tangents of two parabolas with parallel axes, which are each harmonically inscribed to a conic S, centre C; prove that the diameter of S conjugate to CT is parallel to the axes of the parabolas.

Theorem 221 yields an interesting proof of Gaskin's theorem.

Let S_1, S_2 be two circles harmonically circumscribed to a conic Σ.

If they meet at A, B, by Theorem 221, AB and the line at infinity (the join of the circular points) are conjugate lines w.r.t. Σ.

$\therefore$ AB passes through the pole of the line at infinity w.r.t. Σ, *i.e.* the centre of Σ, say O.

$\therefore$ the tangents from O to all circles harmonically circumscribed to Σ are of equal length.

$\therefore$ all circles harmonically circumscribed to Σ are orthogonal to a certain circle σ, centre O.

$\therefore$ the point-circles which are harmonically circumscribed to Σ must lie on σ. Therefore the isotropic lines, forming one of these point-circles at P, are conjugate w.r.t. Σ; therefore the tangents from P to Σ are at right angles and therefore P lies on the director circle of Σ.

$\therefore$ σ must be the director circle of Σ. Q.E.D.

Another method for finishing this proof is given in Ex. 88: and another method of proof in Ex. 89.

88. If in the above proof CA, CB are the semi-axes of Σ, and if T is the pole of AB w.r.t. Σ, prove that the circle through T, touching AB at A, is harmonically circumscribed to Σ, that CT is a tangent to this circle and that its length is $\sqrt{CA^2+CB^2}$.

89. The director circle of Σ meets a circle σ, harmonically circumscribed to Σ, at A; the polar of A w.r.t. Σ cuts Σ at P, Q and σ at B, C; R is the mid-point of PQ; prove that (1) $\{BC; PQ\}$ is harmonic and $P\hat{A}Q = 90°$; (2) $RB.RC = RP^2 = RA^2$; (3) RA touches σ and passes through the centre of Σ; (4) σ is orthogonal to the director circle of Σ.

THEOREM 223 [STEINER'S THEOREM].

A variable conic is inscribed in a given triangle: if the sum of the squares of its axes is constant, the locus of its centre is a circle, whose centre is the orthocentre of the triangle.

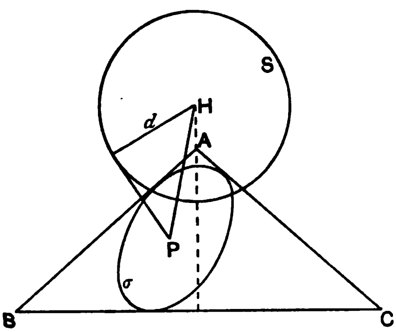

FIG. 169.

Let ABC be the given triangle, and H its orthocentre.

Let c^2 be the constant sum of the squares of the semi-axes of the variable conic, and d the radius of the circle S w.r.t. which the triangle ABC is self-conjugate.

Let P be the centre of one of the inscribed conics, σ.

By Theorem 215, S is orthogonal to the director circle of σ.

But the centre P of σ coincides with the centre of its director circle.

$$\therefore\ PH^2 = c^2 + d^2 = \text{constant}.$$

$\therefore$ the locus of P is a circle, centre H. Q.E.D.

90. A variable rectangular hyperbola touches three fixed lines; find the locus of its centre.

91. Prove that the polar circles of the four triangles, formed by four straight lines, are coaxal, and that their radical axis passes through the mid-points of the joins of opposite vertices of the quadrilateral, formed by the four lines.

THEOREM 224 [CHASLES' THEOREM].

If the sides of a triangle PQR touch a conic S_1, and if Q, R move on conics S_2, S_3 confocal to S_1, then P moves on a conic S_4 also confocal to S_1.

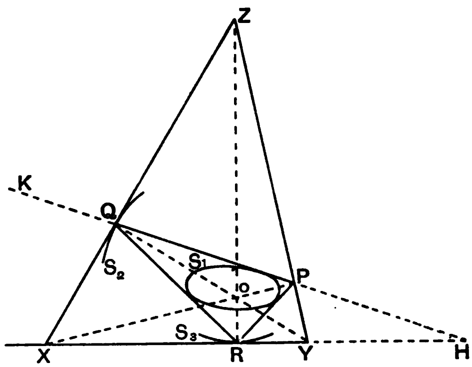

FIG. 170.

Let $P'Q'R'$ be an adjacent position of PQR.

By Theorem 204, P, P', Q, Q', R, R' lie on a conic σ.

Produce PP', QQ', RR' to form the triangle XYZ.

Since in the limit, σ touches the sides of XYZ at P, Q, R, by applying Brianchon's theorem to the hexagon $PPQQRR$, it follows that PX, QY, RZ are concurrent, at O say.

Produce QP to meet XY at H: from the quadrilateral $QPYX$, $\{XY;\ RH\}$ is harmonic, and therefore $Q\{XY;\ RP\}$ is harmonic: produce PQ to K.

By Theorem 199 (2), page 322, since ZQX touches a conic S_2, confocal to S_1, $X\hat{Q}R = Z\hat{Q}P = X\hat{Q}K$.

But $Q\{XY;\ RP\} = -1 = Q\{XY;\ RK\}$;

$$\therefore\ X\hat{Q}Y = 90^\circ;$$

$$\therefore\ QY \text{ is perpendicular to } XZ.$$

Similarly, since R moves on a conic S_3, confocal to S_1,

$$RZ \text{ is perpendicular to } XY;$$

$$\therefore\ O \text{ is the orthocentre of } XYZ;$$

$$\therefore\ PX \text{ is perpendicular to } YZ;$$

$\therefore$ by similar reasoning, P moves on a conic, confocal to S_1.

Q.E.D.

Corollary.

If a quadrilateral circumscribes a conic S, and if three of its vertices trace out conics confocal to S, then every vertex traces out a conic confocal to S, provided that each side passes through at least one of the given vertices.

[Let A, A'; B, B'; C, C' be the pairs of opposite vertices: let A, A', B be the given vertices: apply Theorem 224 to the circumscribing triangles which have AB and $A'B$ as bases.]

92. Prove the Corollary of Theorem 224.

93. Extend Theorem 224 to the case of a polygon circumscribing a conic.

94. Prove Poncelet's theorem: that if a variable triangle ABC is inscribed in a given circle S, and if AB, AC touch fixed circles coaxal with S, then BC touches a circle coaxal with S.

95. S_1, S_2, S_3 are three conics inscribed in a given quadrilateral; a variable triangle ABC circumscribes S_1; B lies on S_2; C lies on S_3; prove that the locus of A is a conic inscribed in the given quadrilateral.

96. Generalise Ex. 95.

97. S_1, S_2, S_3 are three conics passing through four common points; a variable triangle ABC is inscribed in S_1; AB touches S_2; AC touches S_3; find the envelope of BC.

98. Generalise Ex. 97.

THEOREM 225.

The cross ratio of the pencil, formed by the polars of any point H w.r.t. four conics of a pencil of conics through four fixed points, is independent of the position of H.

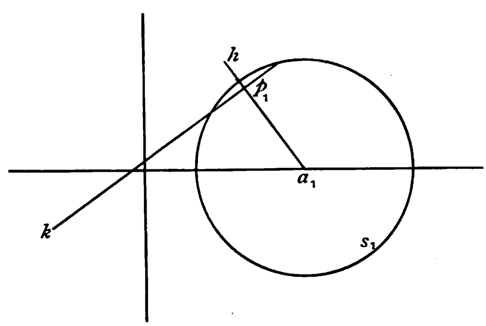

FIG. 171.

Project the pencil of conics into a system of coaxal circles.

Let a_1, a_2, a_3, a_4 be the centres of the circles s_1, s_2, s_3, s_4. Let the polars of h w.r.t. the circles concur at the point k: and let ha_1, ha_2, ha_3, ha_4 meet the polars of k w.r.t. the circles s_1, s_2, s_3, s_4 at p_1, p_2, p_3, p_4.

Then

$$k\{p_1p_2p_3p_4\} = h\{p_1p_2p_3p_4\}, \text{ since the pencils are equiangular,}$$
$$= \{a_1a_2a_3a_4\}$$
$$= \text{constant}.$$

∴ in the original figure, the pencil of polars is of constant cross ratio. Q.E.D.

Corollary.

The pencil formed by the polars of any point H w.r.t. any number of conics of a pencil through four fixed points is homographic with the pencil of polars of any other point H'.

Another method of proof is indicated in Ex. 99.

99. Prove Theorem 225 without projection, by using Theorem 192.

100. Four conics pass through four common points A, B, C, D; prove that the cross ratio of the tangents at A to the conics is equal to the cross ratio of the tangents at B.

101. Prove that the range formed by the poles of a line l_1 w.r.t. a system of conics touching four fixed lines is homographic with the ranges formed by the poles of any other line l_2 w.r.t. the conics.

102. Four conics are inscribed in the quadrilateral $ABCD$; prove that the cross ratio of their points of contact with AB is equal to the cross ratio of their points of contact with BC.

103. $ABCD$ is a given quadrangle; find the locus of the centre of a conic, having ABC as a self-conjugate triangle and D a point on its director circle.

104. Construct the asymptotes of a hyperbola, given its centre and a self-conjugate triangle.

105. Find the locus of centres of hyperbolas which pass through two fixed points and have their asymptotes parallel to two fixed lines.

106. The conic S_1 is harmonically circumscribed to the conic S_2. If a tangent to S_2 cuts S_1 at A, B, prove that the tangents at A, B to S_1 are conjugate w.r.t. S_2.

107. The sides BC, CA, AB of a triangle touch a conic at D, E, F respectively; a tangent at any point P on the conic cuts BC, CA, AB, EF, FD, DE at a, b, c, d, e, f; prove that $\{abcP\}=\{defP\}$.

108. A_1, A_2, A_3, A_4 are four points; S_1, S_2, S_3, S_4 are four conics with their centres at one of the four points, and having the triangle formed by the other three as a self-conjugate triangle; prove that S_1, S_2, S_3, S_4 are homothetic to each other and to the conic-locus of the centres of all conics through A_1, A_2, A_3, A_4.

109. The orthocentre of a variable triangle, circumscribing a given conic, is at a focus of the conic; prove that the polar circle and the circumcircle of the triangle are fixed.

110. S is a point on the polar circle of the triangle ABC; prove that a directrix of the conic, which is inscribed in ABC, and has one focus at S, passes through the orthocentre of ABC.

111. A, A'; B, B' are two pairs of opposite vertices of a quadrilateral, circumscribing a conic; PQ is any chord of the conic, concurrent with AA', BB'; prove that A, A', B, B', P, Q lie on a conic.

112. Prove that the axes of a conic, inscribed in a fixed quadrilateral, the sides of which touch a circle, envelope a parabola, whose directrix is the line through the mid-points of the joins of opposite vertices of the quadrilateral.

113. The three meets of the pairs of chords joining four points on a rectangular hyperbola are concyclic with the centre of the hyperbola.

114. Two concentric coaxal ellipses are such that triangles can be inscribed in one and circumscribed to the other. ACA', BCB' are the principal axes of the larger and aCa', bCb' of the smaller. Prove that the tangents at a, b meet on a side of the parallelogram $ABA'B'$.

115. A variable triangle is inscribed in a fixed circle and circumscribes a fixed parabola; prove that the locus of its centroid is a straight line.

116. A variable ellipse is inscribed in a triangle ABC and passes through the circumcentre of ABC; prove that its director circle touches the circumcircle and the nine-point circle of ABC.

Hence find the locus of the centre of the ellipse.

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